AD 616517

Therm Advanced Research, Inc.

100 HUDSON CIRCLE • ITHACA, NEW YORK

BLANK PAGES
IN THIS
DOCUMENT
WERE NOT
FILMED

ON THE HYDRODYNAMIC THEORY OF WATER-EXIT AND -ENTRY

by

John P. Moran

TAR-TR 6501 March 1965

Submitted to

Fluid Dynamics Branch
Office of Naval Research

In Partial Fulfillment of Contract Nonr-4438(00)

Approved:

A. Ritter

A. Ritter
President

FOREWORD

Since January 1961, personnel of Therm Advanced Research have carried out research on the hydrodynamics of water-exit and -entry under sponsorship of the Fluid Dynamics Branch of the Office of Naval Research. This report is intended to summarize those studies and to show how they complement previous results. Beyond this, it reviews the present state of the art of predicting the hydrodynamic loads on a body crossing a water surface, and suggests directions for future research on this still-unsolved problem.

The report has benefitted greatly from the editorial comments of G. R. Hough and A. Ritter. The author is also pleased to acknowledge the advice and encouragement of those who guided the research which is summarized herein, namely D. E. Ordway, W. R. Sears, and H. S. Tan.

ABSTRACT

The mathematical theory of water-exit and -entry is critically reviewed. A detailed examination of each of the principal methods of analysis available shows that none of them yields even a uniformly valid approximation to the solution during surface crossing. It is concluded that, due to mathematical difficulties inherent in the problem, the best hope for obtaining a reliable estimate of the loads felt in crossing lies with a numerical analysis. Specific recommendations are made as to the formulation of practical numerical methods.

TABLE OF CONTENTS

PRINCIPAL NOMENCLATURE

b	half-width of plate used in flat-plate fitting technique
c	sound speed in water
$\underline{F}$	force on body
g	acceleration due to gravity
h	penetration below undisturbed free surface of lowest point of body
ℓ	body length or other pertinent dimension
m	added mass
$\underline{n}$	unit vector normal to boundary surface; directed into fluid
p	pressure
p_s	pressure on free surface
$\underline{q}$	fluid velocity
$R(x^*)$	radius or half-width of body
$S(x^*)$	body cross-sectional area
t	time
U	body speed

u,v	components of fluid velocity in x and r directions, respectively
$w(z)$	complex potential
$X(r^*)$	displacement of body surface above lowest point of body
x,r	space-fixed cylindrical or Cartesian coordinates
x^*,r^*	body-fixed cylindrical or Cartesian coordinates
z	complex position in two-dimensional problems
β	deadrise angle of wedge or cone; angle between generators and horizontal
$\Delta(r,h)$	displacement of free surface above undisturbed position
δ	density ratio across free surface; ρ^+/ρ^-
ρ	fluid density
ρ^+,ρ^-	density above and below free surface, respectively
ϕ	velocity potential
ϕ^+,ϕ^-	potential above and below free surface, respectively
τ	body thickness ratio; maximum diameter or width over length

ON THE HYDRODYNAMIC THEORY OF WATER-EXIT AND -ENTRY

INTRODUCTION

The original motivation for studies of water-exit and -entry was the need to estimate the impact loads felt in seaplane landings. During World War II, the use of airborne torpedoes further stimulated interest in water-entry problems. Still more recently, the desire to operate ships at higher speeds and in rougher seas has introduced the ship-slamming problem to hydrodynamicists, while the advent of underwater-launched missiles has made the water-exit problem of equal interest.

Such problems have attracted quite a number of investigators, whose contributions are widely scattered through the literature and subliterature. Fortunately for current workers in these areas, water-entry theory has been the subject of two fairly recent reviews. Szebehley's (1959) is almost completely nonmathematical, but gives an extensive list of references to both theory and experiment. Chu & Abramson (1959) go into some of the details of the main theories, and make some interesting comparisons between theory and experiment.

The present report is also intended to review critically the available theories of water-entry and -exit. Like our predecessors, our main purpose is to suggest directions for future work. Here, however, considerably more attention is given to the mathematical details of the various theories, and considerably less to the experimental literature. It is not that we distrust the experimental results, although, to be sure, the transient nature of the subject phenomena makes it difficult to obtain reliable data. Rather, it is because our greater emphasis on the mathematics exposes the inadequacies of the available theories in itself, so that detailed comparison with experiments is unnecessary. Our approach has the added advantage of preventing us from accepting an erroneous theory which, fortuitously, happens to agree with some experimental data.

In Chapter One, the assumptions usually invoked in analyses of water-exit and -entry are stated, their physical and mathematical significance explained, and some immediate consequences noted. Attempts to solve the problem formulated in Chapter One are classified, described, criticized, and compared in Chapter Two, while the effects of some of the factors neglected in the "conventional" formulation are discussed in Chapter Three. This is followed by a Summary and Conclusions section, which contains suggestions for future research.

CHAPTER ONE

FORMULATION AND GENERAL RESULTS

This initial chapter is concerned with the mathematical formulation of a "typical" surface-crossing problem. The physical and mathematical significance of the assumptions introduced is explained in detail, as are the role of the added-mass concept in computing the hydrodynamic forces acting on the body and the direct relation between entry and exit problems which follows from the reversibility of the flow.

1.1 Scope, Coordinates, and Nomenclature

As indicated in the Introduction, a number of physical situations qualify as water-exit or -entry problems. For simplicity, we shall confine most of our formal considerations to a limited class of surface-crossing problems, which, however, exhibits most of the mathematical difficulties of the general problem.

Specifically, we define our basic physical situation to be that depicted in Fig. 1 . A body of revolution or a symmetric two-dimensional airfoil is in vertical constant-speed axial motion into a fluid otherwise at rest. The fluid is bounded above by a nearly horizontal surface, which is free to distort under the influence of the body's motion.

Two sets of coordinates shall be employed. The space-fixed (x,r) system has origin at the undisturbed position of the free surface, while the body-fixed (x^*,r^*) system has origin at the body nose. In three-dimensional cases, both systems are cylindrical, while in plane situations they are Cartesian.

The two systems are connected by

$$x^* = x + h \ , \qquad r^* = r \tag{1}$$

where h is the displacement below the undisturbed free surface (the depth of submergence) of the lower end of the body. As is convenient, we shall use h as our time variable. In terms of h, the time derivative is

$$\frac{\partial}{\partial t} = U \frac{\partial}{\partial h} \tag{2}$$

where U is the body speed, defined positive downward.

In these coordinates, the free surface is located at $x = \Delta(r,h)$ and the body surface at $r^* = R(x^*)$ or $x^* = X(r^*)$.

1.2 Assumptions

To simplify the mathematical problem of determining the flow about the body, most (but not all) previous investigators have conducted their studies under the following assumptions.

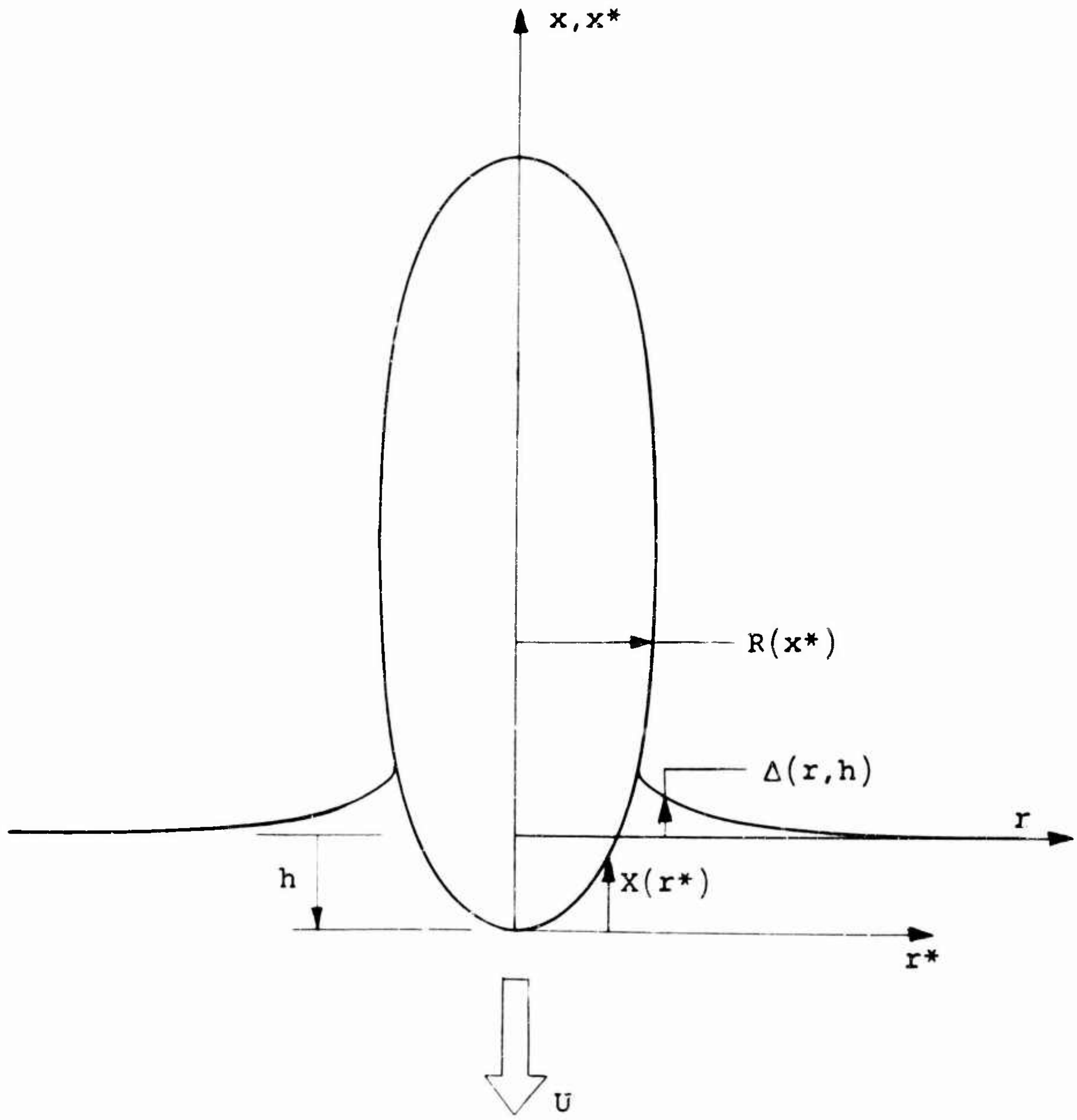


FIGURE 1

COORDINATES AND NOMENCLATURE

1. Irrotational flow. This is justified by the expectation that viscous effects on the pressure felt by the body are generally small, and that the variation in the inviscid forces is more important during surface crossing than the variation of viscous drag. From this assumption, there exists a velocity potential ϕ, which may be regarded as the principal unknown of the problem, since, by definition, its gradient yields the velocity $\underline{q}$;

$$\underline{q} = \nabla\phi \tag{3}$$

2. Incompressible flow. Since exit and entry speeds are usually well below the speed of sound in water, the compressibility of the water is usually neglected. Applying this assumption to the continuity equation, we find that ϕ is governed by Laplace's equation,

$$\nabla^2\phi = 0 \tag{4}$$

This must be satisfied everywhere in the flow field outside the body, except on the air-water interface, across which we anticipate discontinuities in the potential and in its derivatives. The pressure p may now be related to the potential through Bernoulli's equation

$$p = p_s - \rho g x - \rho U \phi_h - \frac{1}{2}\rho(\nabla\phi)^2 \tag{5}$$

Here p_s is the pressure on the (undisturbed) free surface far from the body, ρ is the fluid density, g is the acceleration due to gravity, and the subscripts indicate partial differentiation.

3. Zero air density and surface tension. These factors are generally felt to be unimportant. Their neglect is not essential for an analytic solution, but does simplify matters somewhat. With these assumptions, only the flow in the region below the free surface is of interest, and the dynamic free-surface boundary condition is that the pressure be constant on the free-surface. The kinematic free-surface boundary condition, that the components of the fluid velocity and the surface velocity along the normal to the free surface be equal, may be written

$$U \Delta_h n_x - \phi_n = 0 \qquad \text{on} \quad x = \Delta(r,h) \tag{6}$$

where $\underline{n}$ is the unit normal vector directed out of the flow field, and n_x is its x-component.

4. Zero gravity (infinite Froude number). The justification and motivation for this assumption are the same as for the one preceding. The dynamic free-surface condition

now takes the form

$$U \phi_h + \frac{1}{2} (\nabla\phi)^2 = 0 \qquad \text{on} \quad x = \Delta(r,h) \tag{7}$$

5. <u>No cavitation.</u> There is no real justification for this assumption, except in the initial impact phase of entry problems, or in the relatively uninteresting deep submergence phase of the motion. The neglect of cavitation is, in fact, one of the major defects of water-exit and -entry theory. Unfortunately, present techniques are incapable of coping with two free surfaces in an unsteady-flow problem.

In the absence of cavitation, the only body boundary condition to be satisfied is that the flow be tangent to the given body surface:

$$U n_x + \phi_n = 0 \qquad \text{on} \quad r^* = R(x^*) \tag{8}$$

To complete the formulation, we need the condition of no disturbance far from the body,

$$\phi \to 0 \qquad \text{as} \quad [(x+h)^2 + r^2] \to \infty \tag{9}$$

and the condition that the free surface be initially undisturbed,

$$\Delta = 0 \qquad \text{for} \quad h \leq 0 \tag{10}$$

$$\phi = 0 \qquad \text{on} \quad x = 0 \qquad \text{for} \quad h \leq 0 \tag{11}$$

The problem may now be stated as the determination of the potential so as to satisfy Eqs. (4) and (6)-(11). Once ϕ is known, the pressure distribution on the body can be calculated from Eq. (5). The net force acting on the body can then be found by integrating over the body surface pressure distribution.

1.3 The Added-Mass Concept

An alternative method of computing the force, which is much simpler under certain approximations, is the use of added-mass concepts, which were first applied to water impact by von Kármán (1929). Specifically, letting F_x be the hydrodynamic contribution to the upward force on the body, we have

$$F_x = U \frac{d}{dh} mU \tag{12}$$

where m, the added mass, is given by

$$m = -\frac{\rho}{U} \int_{\Sigma_B+\Sigma_F} \phi \, n_x \, d\sigma \tag{13}$$

Here Σ_B is the wetted portion of the body surface, Σ_F is the free surface, and n_x is the x-component of the unit vector normal to the integration surface, directed out of the flow field.

Equation (13) is well-known; see, e.g., Shiffman & Spencer (1951) for a derivation from momentum principles. For completeness, we present here a derivation in the manner of Landweber & Yih (1956), but specialized to the water-entry problem.

Using Eq. (5), with $g = 0$, we integrate over the body-surface pressure distribution to obtain the net upward force on the body in the form

$$F_x = -\rho \int_{\Sigma_B+\Sigma_F} \left[U\phi_h + \frac{1}{2}(\nabla\phi)^2\right] n_x \, d\sigma \tag{14}$$

Here Σ_B is the submerged portion of the body surface, and Σ_F the free surface. Permission to extend the integration over Σ_F follows from the dynamic free-surface boundary condition (7).

Defining $R_I(t)$ as the radius to the intersection of the body and the free surface, and assuming for convenience that X and Δ are single-valued functions of r (what follows is easily generalized to the case in which they are not), we may write

$$\frac{1}{2\pi} \int_{\Sigma_B+\Sigma_F} \phi \, n_x \, d\sigma = \int_0^{R_I} \phi(X(r)-h,r,t) \; r \, dr$$

$$+ \int_{R_I}^{\infty} \phi(\Delta(r,t),r,t) \; r \, dr \tag{15}$$

Then noting $\Delta(R_I) = X(R_I) - h$, we find that

$$\frac{d}{dh} \int_{\Sigma_B+\Sigma_F} \phi \, n_x \, d\sigma = \int_{\Sigma_B+\Sigma_F} \phi_h \, n_x \, d\sigma - \int_{\Sigma_B} \phi_x \, n_x \, d\sigma$$

$$+ \Delta_h \int_{\Sigma} \phi_x \, n_x \, d\sigma \tag{16}$$

From equations (6), (8), and (16), we thus obtain

$$\int_{\Sigma_B+\Sigma_F} \phi_h \, n_x \, d\sigma = \frac{d}{dh} \int_{\Sigma_B+\Sigma_F} \phi \, n_x \, d\sigma - \frac{1}{U} \int_{\Sigma_B+\Sigma_F} \phi_n \, \phi_x \, d\sigma \tag{17}$$

Now define Σ_∞ as the submerged portion of the surface of a large sphere centered at $(0,0)$. As the radius of this sphere approaches infinity,

$$\int_{\Sigma_\infty} (\nabla\phi)^2 \, n_x \, d\sigma \to 0 \tag{18}$$

assuming that ϕ decays at least as fast as the potential of a point source. Then, from Gauss's theorem and equations (3) and (18),

$$\int_{\Sigma_B+\Sigma_F} (\nabla\phi)^2 \, n_x \, d\sigma = \int_V \frac{\partial q^2}{\partial x} \, d\tau \tag{19}$$

where V is the volume beneath the free surface and the submerged portion of the body surface, and $\mathbf{q}$ is the fluid velocity. Using Cartesian tensor notation for convenience, we have

$$\frac{\partial}{\partial x_i} q_j q_j = 2 q_j \frac{\partial q_j}{\partial x_i} = 2 q_j \frac{\partial q_i}{\partial x_j} = 2 \left[q_j \frac{\partial q_i}{\partial x_j} + q_i \frac{\partial q_j}{\partial x_j} \right] \tag{20}$$

where the second equality follows from irrotationality, and the third from continuity. Substituting (20) into (19) and again using Gauss's theorem and equation (3), we get

$$\int_{\Sigma_B+\Sigma_F} (\nabla\phi)^2 \, n_x \, d\sigma = 2 \int_{\Sigma_B+\Sigma_F} \phi_x \, \phi_n \, d\sigma \tag{21}$$

Equation (13) then follows from (14), (17), and (21).

1.4 Relations Between Exit and Entry Problems

It is of interest to note that, under the above assumptions, the fluid motion under study is reversible. That is, given the shape of the body, its location and orientation with respect to the surface, and its speed, the potential and velocity fields appropriate for an entering body differ only in sign from those appropriate for an exiting body. The pressure field, and hence the loading on the body, is independent of the direction of motion.

To show this, consider an exit problem in which the speed of the body and its orientation in space-fixed coordinates are the same as in the basic entry problem. The formulation of § 1.2 applies equally well to both problems. Thus the equations governing $\bar{\Phi}$ and $\bar{\Delta}$, the potential and free-surface distortion for the exit problem, are obtained by letting $\phi \to \bar{\Phi}$, $\Delta \to \bar{\Delta}$, and $U \to -|U|$ in (4) and (6)-(9), while (10) and (11) must be modified to

$$\bar{\Delta} \to 0 \qquad \text{as} \quad h \to \infty \tag{22}$$

$$\bar{\Phi} \to 0 \qquad \text{on} \quad x = 0 \quad \text{as} \quad h \to \infty \tag{23}$$

But, except for these initial conditions, it is found that $(-\bar{\Phi},\bar{\Delta})$ satisfy the equations which govern the solution (ϕ,Δ) of the relation entry problem. This suggests that

$$\bar{\Phi}(x,r,h) = -\phi(x,r,h) \ , \quad \bar{\Delta}(r,h) = \Delta(r,h) \tag{24}$$

from which the above statements on reversibility would follow directly.

To prove (24), it is sufficient to show that equations (10) and (11) can be replaced in the formulation of the entry problem by (22) and (23). While it is clear that the free surface is undisturbed after the entering body has penetrated infinitely far into the water, it is not so obvious that this condition determines a unique solution. In the case of finite Froude number, in fact, the condition is insufficient, since a backward solution of the entry problem could not recover the Cauchy-Poisson type of wave motions which are set up during broach (Moran 1962b).

The requisite uniqueness proof has been supplied by Moran & Kerney (1964) under the additional assumption that the free surface is only slightly disturbed, which they used to set up a successive-approximation procedure based on expansion of the free-surface boundary conditions in powers of a small parameter. This small-perturbation assumption is not uniformly valid when the body is close to or is broaching the surface. However, it does hold far enough from the body, and it does permit identification of the irreversibility of finite-Froude-number situations. Thus there is no reason to doubt the reversibility even when the expansions made in its derivation fail to converge. While this is somewhat conjectural, practically all analyses are conducted under these small-disturbance approximations, so that their results, at least, are certainly reversible.

Of course, almost any relaxation of the assumptions set forth in § 1.2 destroys the reversibility. We have already noted this for gravity, while cavitating and viscous flows are obviously irreversible. On the other hand, the inclusion of a finite air density does not destroy the reversibility (Moran & Kerney 1964).

CHAPTER TWO

"CONVENTIONAL" SOLUTIONS

In this chapter, we shall survey solutions of exit and entry problems conducted under the "conventional" set of assumptions set forth in § 1.2. As in all problems involving a free surface, a major obstacle to these solutions is the necessity for satisfying nonlinear conditions on a boundary whose location is unknown a priori, while the fact that the flow is basically unsteady is a further complication. No sufficiently general mathematical procedure for treating such problems exists. Thus, all existing analyses of water-exit and -entry either are numerical or are based on an approximate version of the free-surface boundary conditions.

The various approximate methods of solution are reviewed in §§ 2.1-2.4, while the numerical results are discussed in § 2.5. Where possible, the analyses are evaluated against a hypothetical exact solution as they are introduced. A graphical comparison of the predictions of the analyses is given in § 2.6, along with a qualitative discussion of the degree to which theory agrees with experiment.

2.1 Linearization of Free-Surface Boundary Conditions

The most straightforward approximation to the free-surface boundary conditions is formally set up as follows. The terms involving derivatives of the potential in equations (6) and (7) are expanded in Taylor series about $x = 0$, the undisturbed position of the free surface. The potential and all its derivatives are assumed to be small and of the same order of magnitude in some perturbation parameter, at least near the free surface. Then discarding quadratic terms, we obtain

$$\phi = 0 \qquad \text{on} \quad x = 0 \tag{25}$$

$$U\Delta_h = \phi_x(0,r,t) \tag{26}$$

In deriving (25), we have integrated over h and used (11).

The boundary-value problem for the potential — consisting of equations (4), (8), (9), and (25) — is now linear, and is decoupled from the initial-value problem for the free-surface distortion defined by (10) and (26). This, of course, is an enormous simplification over the exact problem.

Moreover, time does not appear explicitly in the linearized boundary conditions on ϕ, but only implicitly, through the time-dependence of the penetration h (see Fig. 1) of the body into the water. In fact, at any given h, the problem is equivalent to one of steady unbounded flow about a body formed by reflecting the portion of the body beneath the undisturbed free surface about that plane, as shown in Fig. 2. For, from symmetry, the streamlines of the equivalent flow are all normal

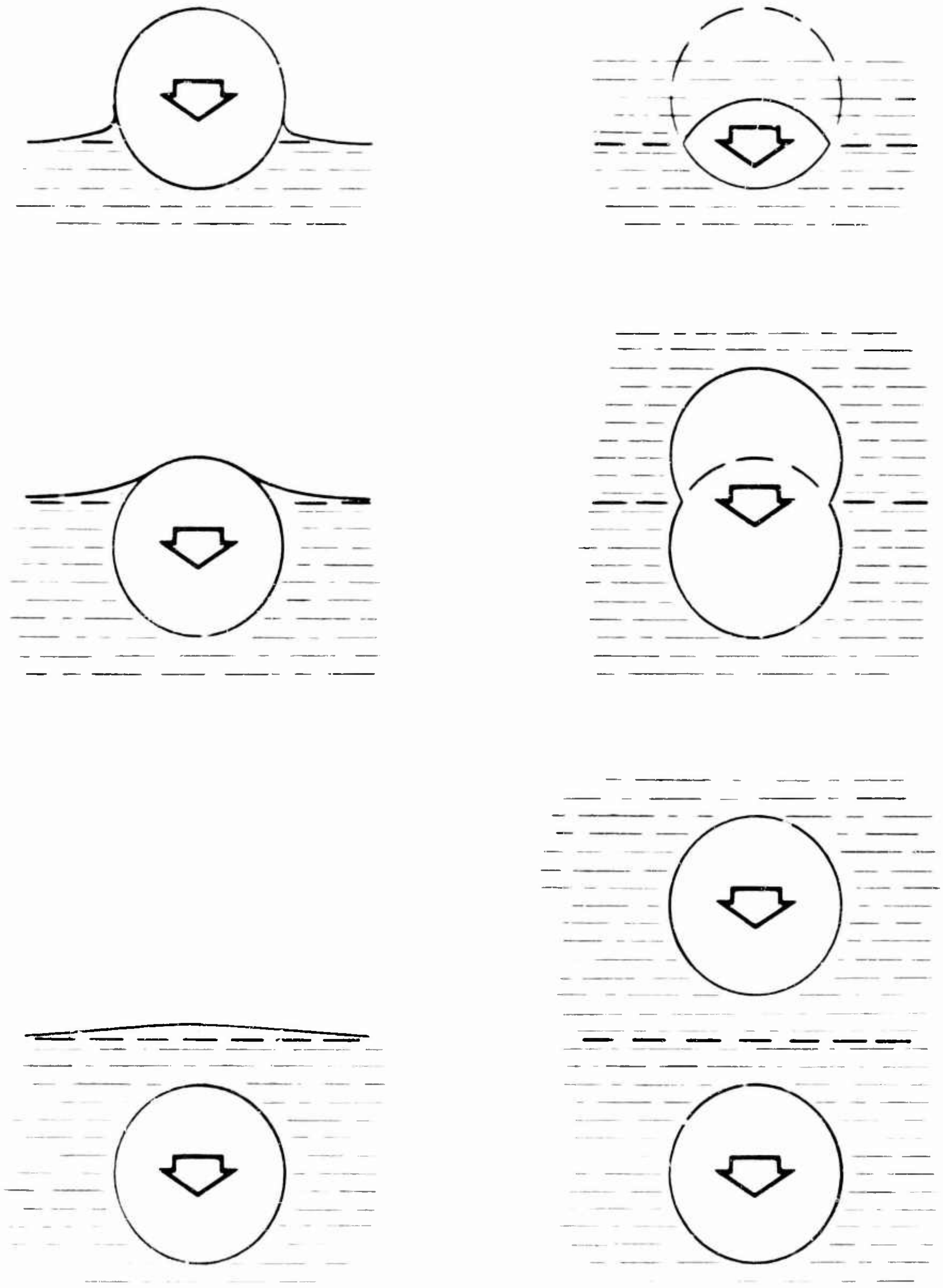


FIGURE 2

EQUIVALENCE OF LINEARIZED WATER ENTRY
WITH PROBLEMS IN UNBOUNDED FLOW
AT VARIOUS STAGES OF PENETRATION

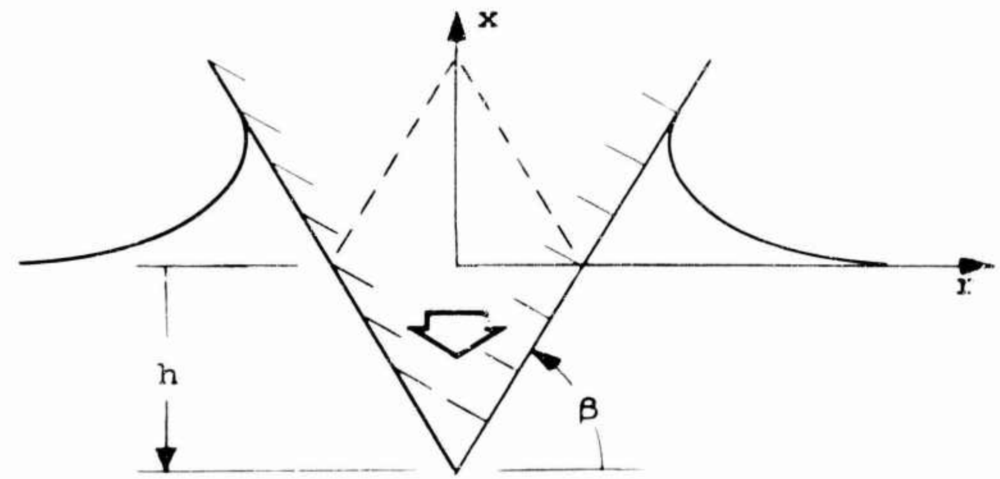
to the $x = 0$ plane, which is then an equipotential, as required. Also, satisfaction of the body boundary condition in the equivalent problem insures satisfaction of equation (8).

As is consistent with the linearization, the integration over the free surface in calculating the added mass from (13) is replaced by an integration over the plane $x = 0$. But, since $\phi = 0$ on $x = 0$ according to the linearized boundary condition (25), this integration does not contribute to m . Moreover, in the equivalent unbounded-flow problem, both ϕ and n_x are antisymmetric about $x = 0$. Therefore, the added mass is half that of the reflected body studied in the equivalent problem.

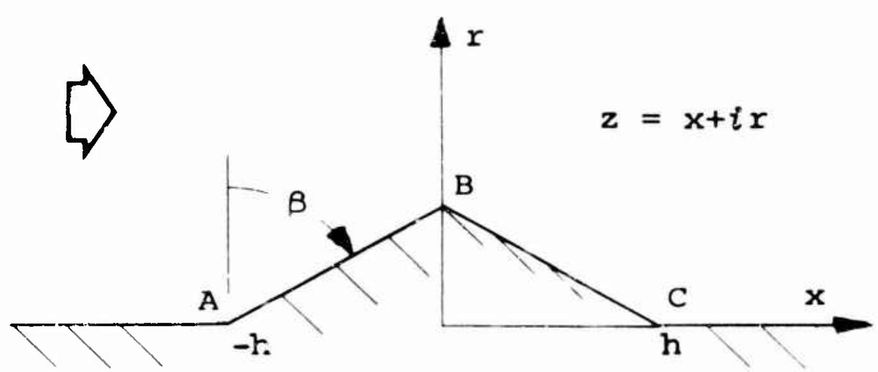
To illustrate, let us consider the vertical symmetric entry of a wedge of deadrise angle β , defined as the angle between a side of the wedge and the horizontal, as shown in Fig. 3. The equivalent unbounded-flow problem, also shown in Fig. 3, is the symmetric flow past a diamond or rhombus, or, what is by symmetry the same thing, the flow past an isosceles triangle sitting on a half-plane. The flow region in this last problem may be mapped onto the upper half of the ζ-plane (say) by a Schwarz-Christoffel mapping (Milne-Thomson 1960), which is given in differential form by

$$\frac{dz}{d\zeta} = A\ \zeta^{1-2\beta/\pi}\ (\zeta^2-1)^{\beta/\pi-1/2} \tag{27}$$

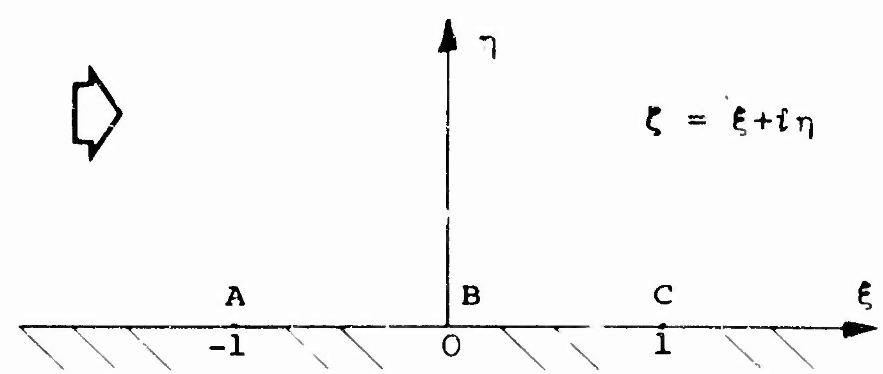
where $z = x + ir$ and the constants of the mapping have been



Physical Problem



Equivalent Unbounded-Flow Problem



Transformed Planc

FIGURE 3

LINEARIZED SOLUTION FOR WATER ENTRY OF A WEDGE chosen so that the corners of the triangle map onto the points -1, 0, and $+1$ in the ζ-plane, as shown in Fig. 3. The complex potential in the ζ-plane is simply

$$w = UA\zeta \tag{28}$$

where the constant has been chosen so that the complex velocity $dw/dz \rightarrow U$ as $z \rightarrow \infty$, after noting from (27) that

$$\lim_{z\to\infty} \frac{dw}{dz} = \frac{1}{A} \lim_{\zeta\to\infty} \frac{dw}{d\zeta} \tag{29}$$

To determine A, we integrate (27) from $\zeta = 0$ to 1, noting that

$$\begin{aligned} z_C - z_B &= \frac{h}{\sin\beta}\, e^{i(\beta-\pi/2)} \\ &= Ae^{i(\beta-\pi/2)} \int_0^1 \xi^{1-2\beta/\pi} \, (1-\xi^2)^{\beta/\pi-1/2} \, d\xi \end{aligned} \tag{30}$$

The integral can be found in tables, and we obtain

$$A = \pi h \left[\Gamma\left(\frac{1}{2}+\frac{\beta}{\pi}\right) \Gamma\left(1-\frac{\beta}{\pi}\right) \sin\beta \right]^{-1} \tag{31}$$

The perturbation velocity potential is, from (28),

$$\phi = U\left[A\, \mathcal{R}e\, \zeta - x\right] \tag{32}$$

in which we have adjusted the usual arbitrary constant so that $\phi = 0$ on $x = 0$ (where ζ is pure imaginary). In principle, the velocity and pressure fields may now be computed by integrating (27) and substituting (32) into (3) and (5). This task is complicated by the fact that the integration of (27) yields z, even on the body, as an incomplete beta function of ζ. However, it is not difficult to obtain the added mass. Substituting (32) into (13), we find

$$m = -2\rho A \int_0^1 \zeta \frac{dr}{d\zeta} d\zeta - 2\rho \int_0^{h \cot \beta} \left[h - r \tan \beta \right] dr$$

$$= \pi \rho h^2 \left\{ \left(\frac{\pi}{2} + \beta \right) \left[\Gamma\left(\frac{1}{2} - \frac{\beta}{\pi} \right) \Gamma\left(1 - \frac{\beta}{\pi} \right) \sin \beta \right]^{-2} - \frac{1}{\pi} \cot \beta \right\} \tag{33}$$

in which we have used (27) and (31) and set $\phi = 0$ on the free surface.

The above results were first given by Lewis (1929) and Taylor (1930) in connection with the problem of a vibrating ship. In analyses directed towards the impact problem, Wagner (1932) quoted equation (33) without derivation, while full details of a solution differeing only slightly from the present one were given by Monaghan (1949) and Karzas (1952).

Taylor also gave the added mass of a two-dimensional symmetrical lens formed by two circular arcs, which figures into the solution for the broadside impact of a circular cylinder. This solution was later reproduced by Fabula (1955).

The most elegant example of this method of analysis is Shiffman & Spencer's (1945a) study of the water-entry of a sphere, which required the solution of the flow about an axisymmetric lens (Shiffman & Spencer 1947). The submerged phase of the water-exit of a sphere was considered by Breslin & Kaplan (1957) and by Woo (1959). However, in view of the reversibility noted in § 1.4, these solutions only duplicate the submerged portion of Shiffman & Spencer's solution.

The small-perturbation assumption on which all these analyses are based is not uniformly valid when the body is close to or is broaching the surface. This is easily seen from the nature of the equivalent unbounded-flow problem, the solution of which is necessarily singular at the sharp corner corresponding to the intersection of the body and the free surface. Now we are accustomed to such singularities in fluid mechanics; in particular, we may mention the leading-edge singularity of thin-airfoil theory, which has a mathematical origin similar to that of the problem under discussion. However, in the airfoil problem, the singularity is simply an exaggeration of the flow around a thin leading edge. In the entry problem, the flow near the point at which the body contacts the free surface is supposed to be parallel to the body surface, so that the singularity of the linearized solution there is more fictitious and hence, presumably, more serious. As we shall see, this defect is shared by virtually all theories of water-exit and -entry, which can, for the most part, be regarded as either improved or simplified versions of the basic linearized theory outlined above.

2.2 Approximation of Body Boundary Condition

It is implied in § 2.1 that the body boundary condition is to be satisfied exactly. For most body shapes, even in unbounded flow, this is not possible without recourse to computer programs, such as those devised by Landweber (1951, 1959) and by Smith & Pierce (1958). Moreover, since the free-surface boundary conditions are satisfied only approximately, a more exact treatment of the body boundary condition is not necessary. Indeed, it may even be deleterious, since, as noted in § 2.1, an exact solution of the equivalent problem sketched in Fig. 2 contains spurious singularities at the intersection of the body and the undisturbed free surface.

2.2.1 Fitting Techniques

In entry problems, the body boundary condition is usually simplified by approximating the wetted portion of the body by some easily analyzed shape. Von Kármán (1929), in his two-dimensional analysis of seaplane-float impact, assumes that the body's penetration into the water is so slight that the body boundary condition can be satisfied on the undisturbed free surface $x = h$. Thus the flow is taken to be that around a flat strip of width equal to the beam $R(h)$ of the float at its intersection with the undisturbed free surface, and the body is said to be "fitted" with a flat plate. From the discussion of § 2.1, the added mass per unit length of the float is then half that of a plate of width $R(h)$ immersed in an unbounded flow directed normal to the plate, or

$$m(h) = \frac{\pi}{2} \rho \, R^2(h) \tag{34}$$

Von Kármán's method was extended to the vertical and oblique entry of bodies of revolution by Plesset (1942), and to three-dimensional shapes approximating seaplane floats or keels by Yu (1945). In the former paper, the flow is taken to be that around an elliptical disk fitted to the intersection of the body with the undisturbed free surface, while the latter uses experimental data on rectangular plates impacting on the surface.

Flat-plate fitting ignores the shape of the submerged portion of the body. In their studies of vertical water-entry of spheres and cones, Shiffman & Spencer (1945a,1951) take the added mass to be half that of an ellipsoid of revolution whose maximum cross-section corresponds with the intersection of the entering body and the undisturbed surface, and whose lower extremity coincides with that of the body. They also tried fitting the submerged portion of the entering sphere with another sphere of radius $R(h)$. Armstrong & Dodd (1957) treated cone entry by fitting a circular-arc spindle to the cone. Trilling (1950b) studied oblique entry of two- and three-dimensional bodies with ellipsoid fitting, while Karzas (1952) and Fabula (1957) used ellipse fitting for vertical two-dimensional entry of wedges and circular cylinders, respectively. As they note, in the approximation under discussion, ellipse fitting and flat-plate fitting give identical results in two dimensions.

2.2.2 Slender-Body Theory

The accuracy of fitting procedures is difficult to assess, though it is certain that the precision worsens with increasing depth of submergence. In dynamical studies, where force data are required through broach and during the submerged phase of the motion, a different approximation to the body boundary condition is preferred, viz., slender-body theory.

To fix ideas, consider the vertical water-entry of a slender body of revolution. We satisfy the Laplace equation automatically by working in terms of its singular fundamental solutions; i.e., sources. In general, such singularities can be distributed only on or within the body surface and on or above the free surface. Symmetry arguments like those of § 2.1 show that the linearized free-surface boundary condition (25) is satisfied if, for each body-bound source, a singularity of equal but opposite strength is positioned at its image in the undisturbed free surface. Since, in formal slender-body theory, the body-bound sources are confined to the body axis, the total singularity system consists of a distribution of sources along the axis of motion. Then, by any one of several techniques familiar from problems in unbounded flow, we find that the body-bound source strength is proportional to the slope of the body mass-sectional area distribution $S(x^*) \equiv \pi R^2(x^*)$, so that the potential may be written

$$\phi = -\frac{U}{4\pi}\int_0^{\ell} S'(\xi^*)\,\left[(x^*-\xi^*)^2 + r^{*2}\right]^{-\frac{1}{2}} d\xi^* + \frac{U}{4\pi}\int_0^{\ell} S'(\xi^*)\,\left[(x^*-2h+\xi^*)^2 + r^{*2}\right]^{-\frac{1}{2}} d\xi^* \qquad (35)$$

where ℓ is the body length. This form is appropriate when the body is completely submerged; when it is not, the upper limits of the integrals should be h.

The axial force on the body can be found either from added-mass concepts or by direct integration of the body surface pressure distribution; no difference in labor is involved. In the case of complete submergence, the result is

$$F_x = \frac{\rho U}{4\pi} \int_0^{\ell} \int_0^{\ell} \frac{S'(x^*)\, S'(\xi^*)\, d\xi^*\, dx^*}{\{x^* + \xi^* - 2h\}^2} \tag{36}$$

Applications of slender-body theory to water entry have been restricted to the "conical-flow" problems of wedge and cone entry. Mackie (1962) studied both cases, including that in which the wedge axis is inclined at a small angle to the vertical. He also considers the possibility that the water surface is not flat, but is wedge-shaped. Coombs (1956) treated a similar problem, entry of a cone into a cone of water, having in mind a "strip theory" of oblique cone entry. That is, the pressure distribution along a generator of the cone is taken to be the same as that along a cone of the same apex angle entering vertically a cone of water whose angle differs from that of the cone by the angle between the given generator and the undisturbed free surface.

Fraenkel's (1958b) analysis of vertical cone entry (into a flat water surface) differs from the formal slender-body theory described here in that he takes into account the discontinuous slope of the reflected body associated with linearization of the free-surface boundary conditions. Using a technique based on the Fourier-transform approach to slender-body theory (Fraenkel 1958a), he obtains a solution which satisfies the body boundary condition to a uniformly valid first approximation, which cannot be said of formal slender-body theory. However, we again note that exact satisfaction of the body boundary condition at the intersection of the body and the free surface may increase the degree to which the free-surface boundary conditions are violated in the same region. Thus Fraenkel's modified slender-body theory is not necessarily more accurate in predicting loads than the more formal theory, though it is certainly more satisfying esthetically.

These applications of slender-body theory to water-entry problems do not exploit its virtue of ready applicability to a wide variety of body shapes. The first such analysis was Breslin's (1958) analysis of vertical symmetric water exit. Much of his work, unfortunately not available in the unclassified literature*, was reproduced independently by Moran (1961). However, both analyses contain non-integrable singularities in the pressure distribution near the stagnation points of round-ended bodies of revolution, such as the ellipsoid, and so predict the axial force to be infinite during the broach phase of the motion.

For bodies whose ends are parabolic (i.e., blunt with finite radius of curvature), Moran (1964b) showed that this anomaly can be eliminated by terminating the source distribution at points midway between the ends of the body and their

---

* Some preliminary results were reported by Breslin & Kaplan (1957).

centers of curvature. That is, the integration limits in equation (35) are modified as follows:

$$0 \to \frac{1}{4\pi} s'(0) \; , \; \ell \to \ell + \frac{1}{4\pi} s'(\ell) \tag{37}$$

This device makes the theory a valid first approximation near the stagnation points when those points are not too close to the free surface. In fact, if the body is completely submerged, such that the depth of submergence $h-\ell$ is large compared to $\tau^2 \ell$, where τ is the body thickness ratio, equations (35) and (37) yield a uniformly valid approximation to the flow field, provided sufficient restrictions are imposed on the body shape to insure the applicability of the line-source-distribution technique. Specifically, to a first approximation, it is sufficient that the first two derivatives of the cross-sectional area distribution $s(x^*)$ be continuous over the length of the body (Moran 1963).

But even when the body shape does not satisfy these requirements, so that source rings must be distributed on the body surface to get a uniformly valid solution, equation (35) is still an accurate approximation in the case of "deep" submergence. In particular, it can be shown that the images of the body-bound ring sources can, for the purpose of computing the effect of the free surface on the body pressure distribution (and hence for computing the net force on the body), be approximated by an axial source distribution identical in form with the second term in (35). However, regardless of body shape, no way has been devised to render slender-body theory uniformly valid when the body is only slightly or incompletely submerged.

Goodman (1960) used slender-body theory to estimate the lateral forces and moments felt by a body undergoing small pitching motions while in vertical or near-vertical exit. Here the body-bound singularities and their images are horizontal doublets on the axis of motion. Since, according to slender-body theory, the lateral force at any point on the body is due only to the local singularity strength, there is no effect of the free surface on the lateral forces and moments while the body is completely submerged. Goodman used this fact to deduce fairly simple formulas for the variation of the stability derivatives during broach. Martin (1960) has considered oblique exit with slender-body theory, but his results are incomplete.

In closing this section, it may be noted that the slender-body approximation may be regarded as a generalization of the fitting technique discussed in § 2.2.1. The fitting body is the streamline shape generated by the axial singularity distribution whose strength is taken from slender-body theory. The degree of approximation between the shapes of this streamline and of the given body is poorest near the intersection of the body with the free surface, but improves with increasing distance from the surface. This is illustrated in Fig. 4, which compares the actual streamline shape generated by the singularity distribution calculated by slender-body theory for the surface crossing of a slender ellipsoid of revolution (Moran 1962a) with the shape of the ellipsoid during the broach phase of the motion. Of course, the agreement is much better when the body is completely submerged.

2.3 Corrections to Linearized Solutions

2.3.1 The Wetting Correction

Attempts to improve the accuracy of the simplified analysis described above date back practically to its conception. In particular, Wagner (1932) called attention to the water which is "piled up" along the sides of an entering body. Intuitively, this could have a substantial effect on the impact force, simply because the wetted area of the body is certainly greater than in the linearized picture, and because of the substantial density difference between air and water.

Accordingly, Wagner proposed to correct for the piled-up water by computing the change in free-surface shape with a linearized theory and then matching the heights above the undisturbed free surface of that surface and of the body at their

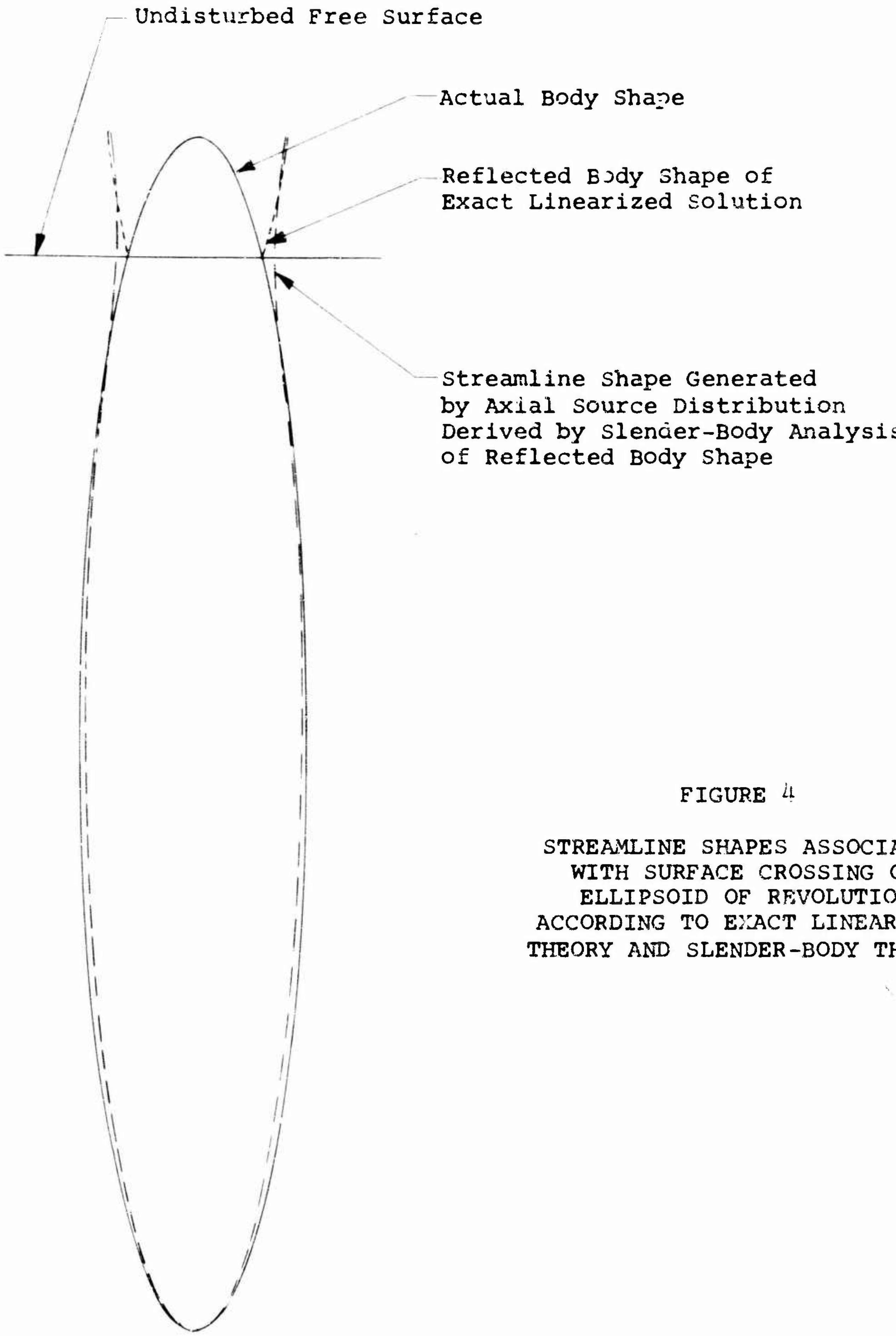


FIGURE 4

STREAMLINE SHAPES ASSOCIATED WITH SURFACE CROSSING OF ELLIPSOID OF REVOLUTION ACCORDING TO EXACT LINEARIZED THEORY AND SLENDER-BODY THEORY

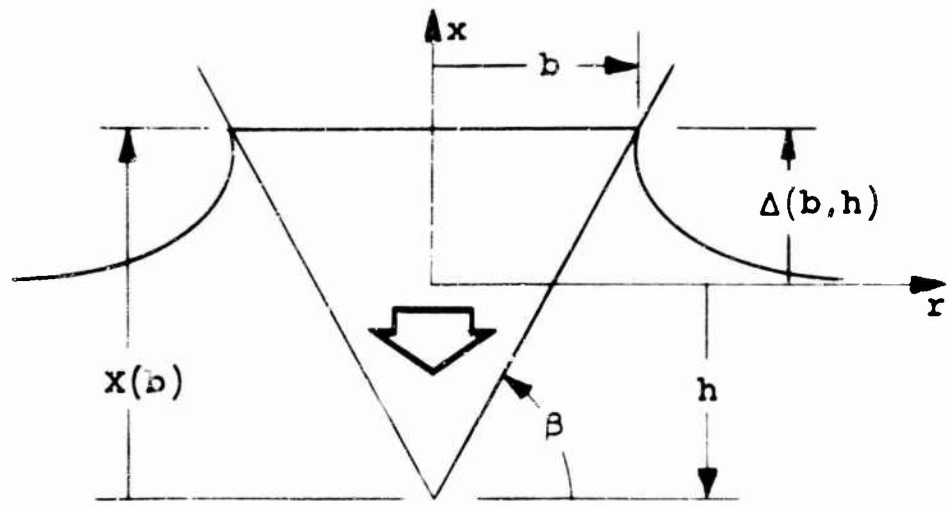


FIGURE 5

WAGNER'S FLAT-PLATE FITTING OF
WEDGE ENTRY WITH WETTING CORRECTION

point of contact. Since this idea has formed the basis for many subsequent analyses, we shall reproduce Wagner's analysis here.

The linearized theory Wagner used to compute the piled-up water is the same as von Kármán's (1929) analysis, which Wagner apparently deduced independently. Thus the entering body is "fitted" with a flat plate which, as shown in Fig. 5, connects the two points at which the body contacts the free surface, $(\Delta(b,h)\ ,\ \pm b)$. Here b , the half-width of the plate, is a function of h to be determined. The potential of the flow about such a plate is given in most texts on hydrodynamics; in the present nomenclature,

$$\phi = U\left[\mathcal{R}e\left\{(x - \Delta(b,h) + ir)^2 + b^2\right\}^{\frac{1}{2}} - x\right] \tag{38}$$

Using (38) to compute ϕ_x on $x = \Delta(b,h)$, we apply the linearized boundary condition (26) to compute the free-surface shape in the form

$$\Delta(r,h) = \int_0^b \left[\frac{r}{(r^2-\zeta^2)^{\frac{1}{2}}} - 1\right] \lambda(\zeta)\, d\zeta \qquad \text{for } r \geq b \tag{39}$$

in which

$$\lambda(c) \equiv \frac{dh}{db} \tag{40}$$

has been introduced to change the integration variable from h to $b(h)$. Now, by definition (see also Fig. 5)

$$X(b) = \Delta(b,h) + h \tag{41}$$

while, from (40)

$$h = \int_0^b \lambda(\zeta)\, d\zeta \tag{42}$$

Thus, letting $r \to b$ in (39) and using (41) and (42), we obtain an integral equation for $\lambda(b)$,

$$X(b) = \int_0^b \frac{b\lambda(\zeta)\, d\zeta}{(b^2-\zeta^2)^{\frac{1}{2}}} \tag{43}$$

Wagner assumes power series for $X(r)$ and $\lambda(b)$,

$$X(r) = \sum_{n=1} A_n r^n \tag{44}$$

$$\lambda(b) = \sum_{n=0} B_n b^n \tag{45}$$

substitutes (44) and (45) into (43), and, after integrating term by term, gets*

$$\sum_{n=1} A_n b^n = \frac{\pi}{2} \sum_{n=1} B_{n-1} \frac{\Gamma(n/2)b^n}{\Gamma(n/2+1/2)} \tag{46}$$

from which the B_n's are easily identified by equating coefficients of like powers of b. Substituting the results into (45), Wagner then finds $h(b)$ from (42).

For example, for the case of a wedge with deadrise angle β, the coefficients in (44) are

$$A_1 = \tan\beta$$

$$A_n = 0 \qquad \text{for } n > 1 \tag{47}$$

*The formula for the general term in the series on the right was written down by Fabula (1957).

and so, from (46),

$$B_0 = \frac{2}{\pi} \tan \beta$$

$$B_n = 0 \qquad \text{for} \quad n > 0 \tag{48}$$

Then, from (39), (45), and (48),

$$\Delta(r,h(c)) = \frac{2}{\pi}\left[r \sin^{-1} \frac{b}{r} - b\right] \tan \beta \tag{49}$$

while (42) yields,

$$b(h) = \frac{\pi}{2} \frac{h}{\tan \beta} = \frac{\pi}{2} R(h) \tag{50}$$

where $R(h)$ is the width of the wedge at its intersection with the undisturbed free surface.

As is formally consistent with a linearized theory, Wagner neglects the difference between the free surface and the equipotential $x = \Delta(b,h)$ in computing the apparent mass from (13), and so arrives at a formula identical in form with von Kármán's (1929) result, equation (34):

$$m(h) = \frac{\pi}{2} \rho\, b^2(h) \tag{51}$$

For the case of a wedge, substitution of (50) into (51) and comparison with (34) shows that Wagner's analysis predicts an impact force $\pi^2/4$ times as great as does von Kármán's theory.

Wagner's treatment of the piled-up water has been widely imitated. Shiffman & Spencer (1945b) employed a similar correction in their study of sphere entry, using the flow due to a circular disk to approximate the upwash on the free surface in computing its rise for the limiting case of zero penetration depth. For deeper penetrations, they determined the "wetting factor" — defined as the ratio of the height of the point of contact above the lowest point on the body to the depth of penetration ($X(b)/h$ in Fig. 5) — from experiments. In their subsequent study of the vertical entry of a cone, Shiffman & Spencer (1951) fitted the wetted portion of the body with half an ellipsoid of revolution, and used its upwash in computing the wetting factor.

Karzas (1952) used the upwash due to a fitted ellipse in treating the wedge. He also suggested a second approximation to the surface rise, in which the full non-linear boundary condition (6) is used, but carried it out only for the case of deadrise angle $\beta = 45°$, in which the difference from the result obtained via the linearized boundary condition (26) was only about 2%. Schnitzer and Hathaway (1953) applied Wagner's flat-plate fitting to treat the entry of an elliptical cylinder, while Fabula (1955) similarly studied broadside impact of circular cylinders. Later (1957)

Fabula used ellipse fitting to find the wetting factor for elliptical and circular cylinders and for wedges, the last application being a recapitulation of Karzas's (1952) work.

Fabula's solution of the integral equation corresponding to (43) is, like Wagner's, carried out by power-series expansion methods. Chu & Abramson (1959) worked out a numerical method for solving the integral equation derived from the ellipse fitting of two-dimensional bodies of arbitrary cross-section, using truncated Fourier series.

Also in his 1957 paper, Fabula computes the water piled up in wedge entry using the formally exact linearized upwash distribution (that due to a diamond). In so doing he corrects an error made by Monaghan (1949) and copied by Bisplinghoff & Doherty (1952), who used an incorrect upwash. Ochi & Bledsoe (1962) used ellipse fitting to calculate the upwash and thence the wetting factor for certain two-dimensional shapes (Lewis forms) which approximate ship hull sections.

The accuracy of these corrections is, of course, difficult to predict. If the approximating body used to generate the upwash distribution is Wagner's flat plate, or the reflected body associated with the exact linearized solution, the distribution contains a spurious singularity at the point of contact of the body with the free surface. Thus, Fabula (1957) claims that, even in cases where the integral equation (cf. (43)) formed from the free-surface boundary condition (26) using the exact linearized upwash may be solved analytically (such as wedge entry), it may be preferable to use the upwash associated with ellipse fitting, which at least is not singular.

2.3.2 The Free-Surface Correction

A correction of a different sort was proposed by Shiffman & Spencer (1945b,1951). This "free-surface correction" is simply the contribution to the apparent mass of the integral over the free surface in equation (13). As in the wetting correction, the potential and free-surface shape employed in the calculations are those given by a solution of the linearized problem (Shiffman & Spencer used ellipse fitting in computing thier corrections), and approximations are made such that a formal first approximation to the integral is obtained. Thus the validity of this correction is subject to the same criticism as is the wetting correction; viz., that the exact first-order solution is spuriously singular near the body. Fabula (1957) has shown that, in the case of wedge entry, use of the exact first-order result (the potential of flow past a diamond) leads to alarmingly large corrections, even for small deadrise angles, while use of flat-plate fitting leads to an infinite correction. He therefore recommends that the free-surface correction not be used, though he does believe in wetting corrections, the major defect of which we have noted have the same origin. The present author would prefer to see both corrections obtained via exact

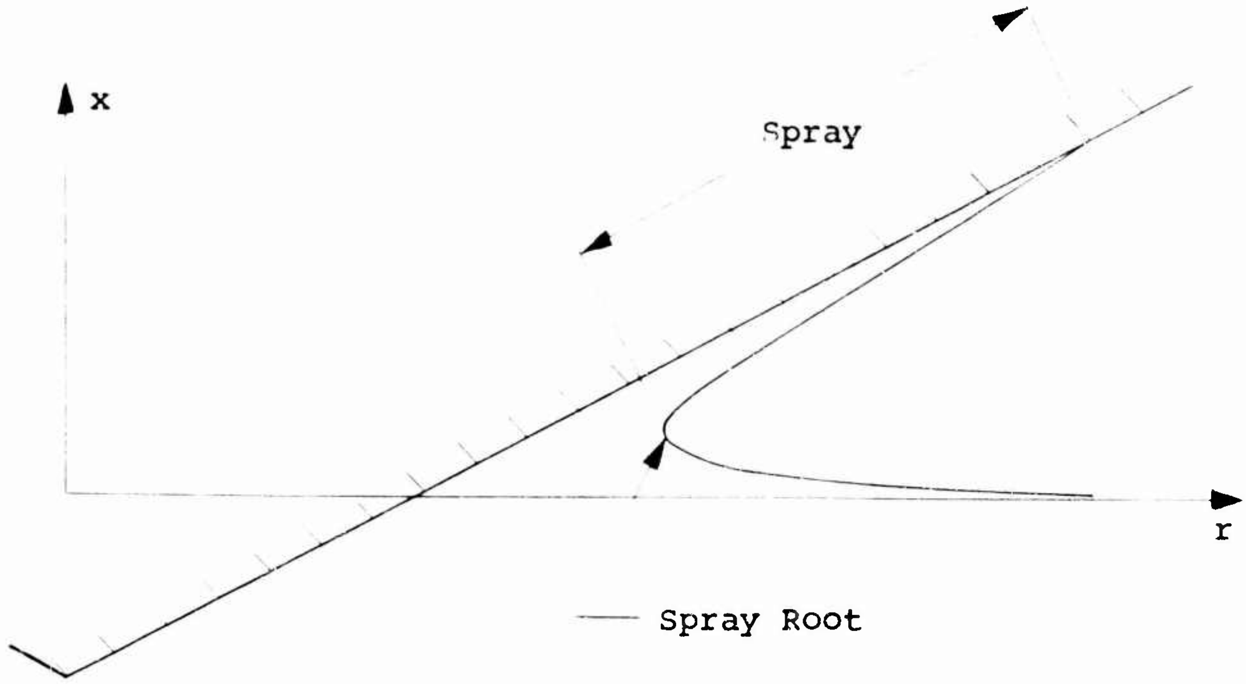


FIGURE 6

SOME NOMENCLATURE FOR THE ENTRY PROBLEM

numerical calculations for a few cases,* but this course also offers difficulty, as we shall see in § 2.5.

2.3.3 The Spray Correction

In an effort to remove the singularity found in linearized theory at the point of contact between the free surface and the body, Wagner (1932), in the same fundamental paper to which we have already referred several times, sought to analyze the flow in this region. Now from experiments — see, for example, the

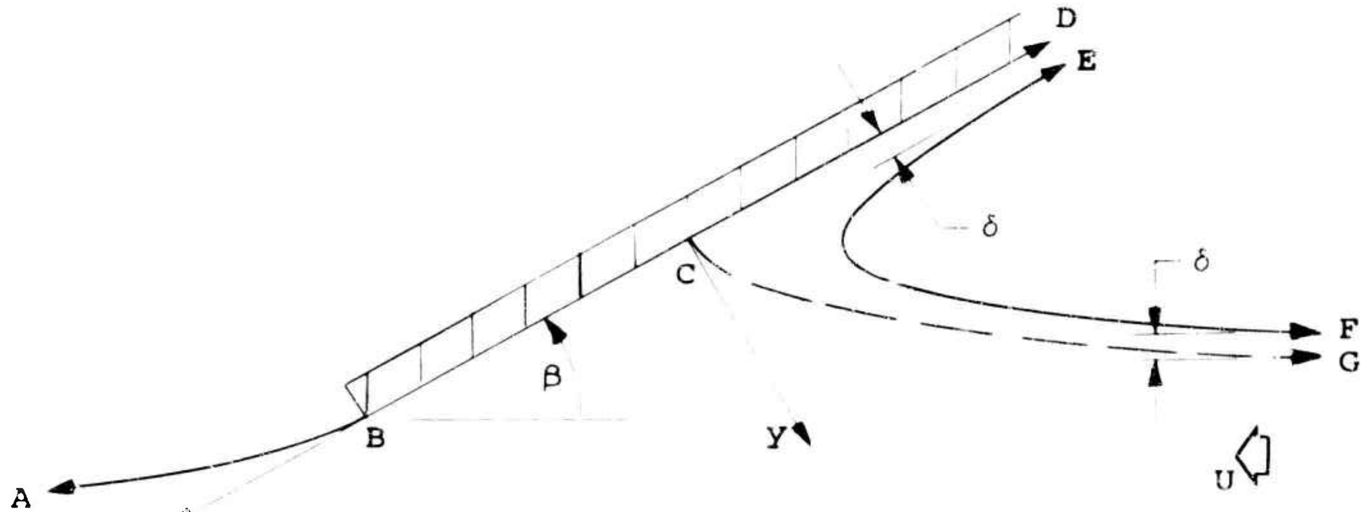
---

*A step in this direction was actually taken by Wagner (1932), who determined a formula for the added mass of a wedge by fitting a curve to three points: for $\beta = 0°$, the result of flat-plate fitting with wetting correction; for $\beta = 18°$, the result of a numerical calculation; and for $\beta = 90°$, the exact linearized solution.

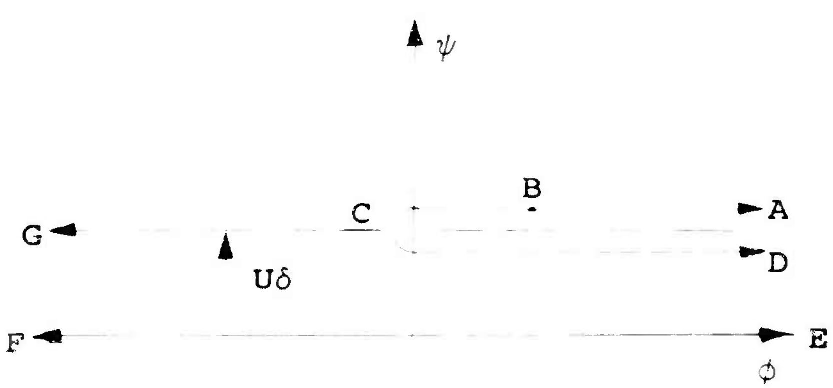
photographs of Bisplinghoff & Doherty (1952) and Borg (1957) — it is known that the free surface is very nearly tangent to the body surface at the contact point, and runs nearly parallel to the body for some distance before breaking out and becoming more nearly horizontal. The thin sheath of water which thus wets the body near the contact point, see Fig. 6, is called the "spray", while the point at which the free surface begins to assume a more horizontal shape is called the "spray root".

Wagner reasoned that the spray should be similar to that formed when a hydrofoil planes or glides at constant speed on the surface, which problem (see Fig. 7) he analyzed by conformal mapping. However, in the planing problem, the spray is of infinite extent, while we expect the surface rise in the entry problem to be finite. Moreover, planing on an infinite body of water is not a properly posed problem in the absence of gravity, in the sense that it is impossible to find a solution in which the depth of immersion of the plate is finite; the free surface far from the plate is a logarithmically infinite distance below it.* This

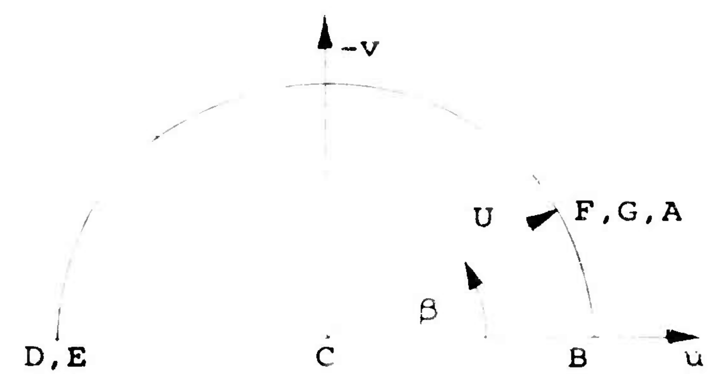
*It is easy to show that this is true whether the plate is supposed to be infinitely long, as it was by Wagner (1932) so as to keep the spray parallel to the gliding plate and hence model the wedge-entry problem, or whether the plate length is allowed to be finite, as it was by Green (1936b) (see also Milne-Thomson 1960). The phenomenon follows from the $1/r$ decay of the velocity field due to the plate (as for a vortex); the kinematic free-surface boundary condition for steady flow, $U\Delta_r(r) \approx u(0,r)$; and the simplicity of the image system in the case of infinite Froude number. For a study of planing at large but finite Froude number, in which case the difficulty disappears, see Cumberbatch (1958).



Physical Plane: $z = x+iy$



Complex Potential Plane: $w = \phi+i\psi$



Hodograph Plane: $dw/dz = u-iv$

FIGURE 7

PLANING PROBLEM STUDIED BY WAGNER
AS STEADY-STATE ANALOG TO ENTRY PROBLEM

difficulty led Green (1935, 1936a) and Cooper (1949) to consider planing on water of finite depth and on jets of finite height, respectively, but such analogies are rather flimsy. It may be concluded (as did Cooper) that, while the deficiencies of the linearized theory near the contact point are quite serious, the steady-state approximation has little to offer in the way of a remedy.

2.3.4 Higher-Order Slender-Body Theory

The difficulties encountered in attempts to correct the linearized results during the broach phase of the motion are not encountered in the case of complete submergence. For this case, Moran & Kerney (1964) have developed a second-order slender-body theory by formally expanding the free-surface boundary conditions in the body's diameter-length ratio τ . The solution is quite complex, involving singularity distributions over the undisturbed free surface in even the simplest cases, and so is impractical as an engineering tool. However, through comparison of the first- and second-order results, one may determine precisely how close the body can approach the surface before the linearized theory breaks down. Such information would be useful were it desired to obtain a numerical solution of the exit problem by integrating over time.

2.4 Other Analytical Methods

2.4.1 Chou's Method

A radically different approach was devised by Chou (1946) for the sphere-entry problem, and extended by Fabula & Ruggles (1955) to the entry of a circular cylinder. Basically, the idea is to use the potential of the unbounded flow about a one-parameter family of bodies defined such that, as the parameter increases (say), increasing portions of the lower surfaces of the family body and of the given body match. Thus, Chou uses the family of spherical bowls having the same radius as the given sphere, while Fabula & Ruggles use its two-dimensional analog, the family of sectors of a circle of fixed radius.

To relate the parameter to time, Chou looked at the asymptotic behavior of the solution. He reasoned that the disturbances far from an entering body are sufficiently small to justify linearization of the free-surface boundary conditions there, and so that the analytic continuation of the potential ought to be anti-symmetric about the plane undisturbed position of the free surface far from the body, as is predicted by the linearized theory. Conversely, such a plane of symmetry ought to be defined as the undisturbed free surface. Thus the parameter characterizing the shape of the equivalent body should be related to the depth of penetration h by finding, as a function of that parameter, the displacement of the lowest point of the body below the plane which is normal to the axis of symmetry and about which the unbounded flow is asymptotically symmetric. Specifically, Chou observed that, far from the body, the potential can be approximated to second order in the inverse distance from the body by the potential of a doublet of the correct strength and location. Because of the symmetry of doublet flow, the undisturbed free surface is specified as the plane containing that doublet.

Thus the depth of penetration is found as a function of the parameter characterizing the potential. The pressure can then be computed from equation (5), and the intersection of the body with the free surface located at that point on the body at which $p = p_s$. The force on the body can then be found by integrating over the pressure distribution on the wetted portion of the body.*

To illustrate these concepts, we again consider the wedge-entry problem.** For this case we need the solution for flow past a hollow wedge, or, by symmetry,

---

*Alternatively, one can linearize the dynamical free-surface boundary condition, and so locate the intersection point as that at which the potential has the same value as at infinity on the undisturbed free surface, and use the added-mass concept to calculate the force; i.e., one can regard the free surface as an equipotential rather than an isobar.

**According to Cox & Maccoll (1957), this problem has been worked out with Chou's method by A. Coombs. It may also be noted that a good part of the corresponding solution for the cone-entry problem can be taken from the work of Rich & Karp (1960) on the flow about a finite conical shell.

for flow past a line jutting out from a half-plane, as shown in Fig. 8. The flow region in the physical plane of the equivalent problem can be mapped on to the upper half of the ζ-plane by the Schwarz-Christoffel mapping

$$\frac{dz}{d\zeta} = A(\zeta-b)(\zeta+1)^{\beta/\pi-1/2}(\zeta-1)^{-\beta/\pi-1/2} \tag{52}$$

in which $z = x^* + ir^*$, and the two sides of the vertex have been mapped onto -1 and $+1$ in the ζ-plane, while the edge of the hollow wedge is mapped onto $\zeta = b$ (say). As in § 2.1, the complex potential is readily found in the ζ-plane to be

$$w = UA\zeta \tag{53}$$

Now as $\zeta \to \infty$, from (52) and (53),

$$\frac{1}{U}\frac{dw}{dz} \cong 1 + \left(b - \frac{2\beta}{\pi}\right)\frac{1}{\zeta} + 0\left(\frac{1}{\zeta^2}\right) \tag{54}$$

while $z \sim A\zeta$ + constant. Thus, we must have

$$b = \frac{2\beta}{\pi} \tag{55}$$

for otherwise the complex potential would be source-like at infinity, which we know is not the case.

Using (55) and the fact that $z(1) = 0$, we find that

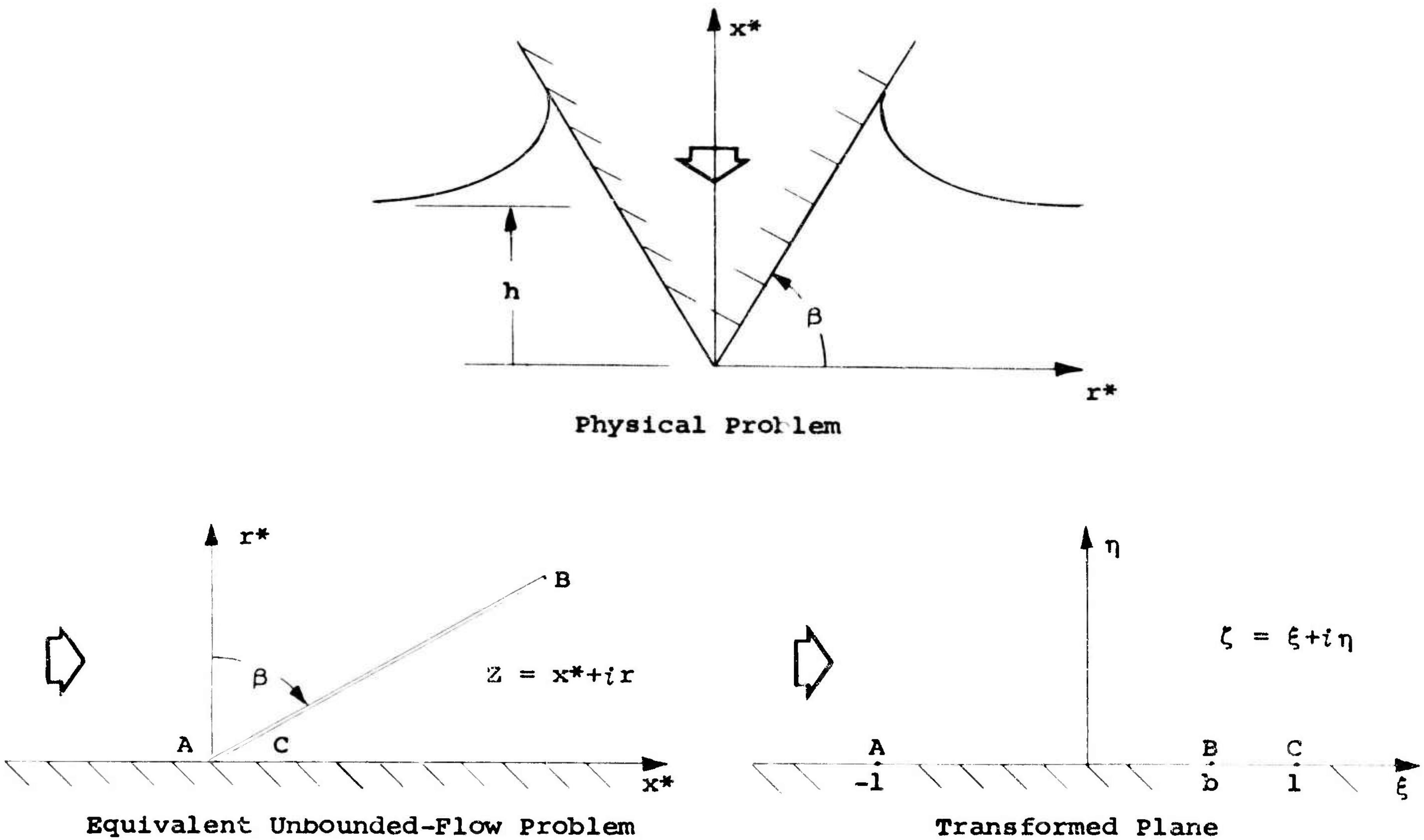


FIGURE 8

CHOU'S SOLUTION FOR WATER ENTRY OF A WEDGE

(52) can be integrated in closed form (!) to yield

$$z = A(\zeta-1)^{1/2-\beta/\pi}\ (\zeta+1)^{1/2+\beta/\pi} \tag{56}$$

The constant A may now be determined quite easily in terms of the length of the side of the wedge, but this is not necessary for our purposes. Rather, we expand (56) for large ζ , and then invert the expansion to obtain, after substituting into (53),

$$\begin{aligned} w &= Uz + \text{const.} + \frac{A^2}{2}\left(1 - \frac{4\beta^2}{\pi^2}\right)\frac{1}{z} + \frac{4}{3}A^3\,\frac{\beta}{\pi}\left(1 - \frac{4\beta^2}{\pi^2}\right)\frac{1}{z^2} + O\left(\frac{1}{z^3}\right) \\ &= Uz + \text{const.} + \frac{A^2}{2}\left(1 - \frac{4\beta^2}{\pi^2}\right)\left(z - \frac{8}{3}\,\frac{\beta}{\pi}\,A\right)^{-1} + O\left(\frac{1}{z^3}\right) \end{aligned} \tag{57}$$

Thus, to second order in the inverse distance from the body, the perturbation potential is due to a doublet at $z = \frac{8}{3}\,\frac{\beta}{\pi}\,A$, and so, according to Chou's method,

$$A = \frac{3\pi}{8\beta}\,h \tag{58}$$

The velocity field may now be easily computed in terms of the parameter ζ from

$$u^* - i v^* = \frac{\partial w}{\partial z} = U\,\frac{3\pi}{8\beta}\,h\,\frac{\partial \zeta}{\partial z} \tag{59}$$

in which we have used (53) and (58) and where the * indicates that the velocity components are measured in body-fixed coordinates. To compute the pressure field, we have the body-fixed version of (5),

$$\frac{1}{\rho}(p-p_s) = -U\phi_h{}^* - \frac{1}{2}(u^{*2} + v^{*2} - U^2) \tag{60}$$

Noting from (56) and (58) that $\zeta = \zeta(z/h)$, so that

$$\frac{\partial\zeta}{\partial h} = -\frac{z}{h}\frac{\partial\zeta}{\partial z} \tag{61}$$

we find (53), (58), and (61) that

$$\frac{\partial\phi^*}{\partial h} = \frac{3\pi}{8\beta} U^2 \mathcal{R}e \left[\zeta - z\frac{\partial\zeta}{\partial z} + \frac{2\beta}{\pi}\right] \tag{62}$$

The constant in the brackets is included to make $\partial\phi/\partial h$ vanish far from the body, after noting that the arbitrary constant in the potential as derived here can be time-dependent. Thus, from (56), (58)-(60) and (62), we get for the pressure distribution on the body surface

$$\frac{p-p_s}{\frac{1}{2}\rho U^2} = 1 + \frac{3\pi}{4\beta}\left[1 - \left(\frac{2\beta}{\pi}\right)^2\right]\left[\frac{2\beta}{\pi} - \zeta\right]^{-1}$$
$$- (1+\zeta)^{1-2\beta/\pi}(1-\zeta)^{1+2\beta/\pi}\left[\frac{2\beta}{\pi} - \zeta\right]^{-2} \tag{63}$$

The point at which the body contacts the free surface is that at which the right side of (63) vanishes, and the force on the body can be found by integrating (63) over the thus determined wetted portion of the body. This we have done numerically for a range of deadrise angle β (see Fig. 17 in § 2.6).

It is seen that Chou's method leads to exact satisfaction of the body and dynamical free-surface boundary conditions, but ignores the kinematic free-surface boundary condition. Thus the method is no more precise than are those discussed previously. Moreover, it is somewhat arbitrary, in that a perturbation in the form of the approximating body in the region above the surface affects the solution. On the other hand, it is no worse than other methods, which also fail to satisfy the free-surface boundary conditions near the body. Further, it has the virtue of yielding realistic (non-singular) pressure distributions, in which respect it is practically unique.

2.4.2 Garabedian's Method

Even more unusual is the approach put forth by Garabedian (1953), which is specialized to oblique wedge-entry. His formulation is in terms of a function W defined on the free surface by*

---

*That such a function exists follows from equation (68).

$$s = t\,W\left(\frac{z}{t}\right) \tag{64}$$

where s is the distance along the free surface and $z = x+ir$, for which he finds an analytic continuation into the whole flow region. He shows that z and the complex potential can be given parametrically in terms of W and dW/dz, and so reduces the problem to one of conformal mapping from the W-plane to the dW/dz-plane.

From this point on, Garabedian's method is essentially an indirect one. To simplify the mapping between the W and dW/dz planes, it is desirable that the contours of the flow domains in these planes be rectilinear (so that they are amenable to Schwartz-Christoffel mappings). Thus Garabedian specifies that W have constant argument on the maps of the wedge, which implies that W be either pure real or pure imaginary on these contours, and leads eventually to the requirement in the physical plane that the free surface separate from the wedge either tangentially (but not so that the wedge and free surface form an upward-pointing cusp, which, Garabedian states, would lead to a singularity) or perpendicularly. He then treats a curious case in which the separation is tangential on one side of the wedge and perpendicular on the other, for which the "physical" plane is as sketched in Fig. 9. The free-surface pressure on one side of the wedge is found to differ from that on the other

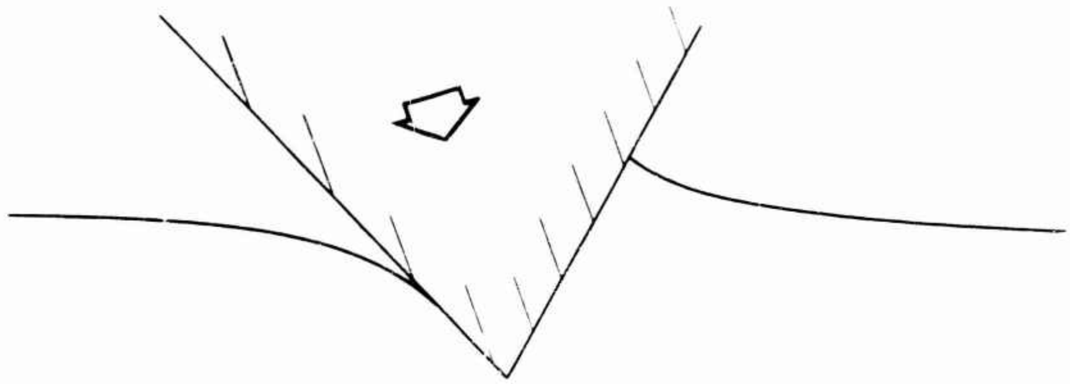


FIGURE 9

ENTRY PROBLEM TREATED BY GARABEDIAN

side, and the pressure in the fluid is non-uniform at $t = 0$. This, of course, does not correspond to any physical situation. We have attempted to apply Garabedian's method to vertical symmetric entry, assuming perpendicular separation on both sides of the wedge, but not with success. The stumbling block we encountered is overcome in Garabedian's case by the fact that dW/dz has a simple zero at the stagnation point, which is separated from the vertex in oblique entry.

We thus conclude that Garabedian's method is incapable of yielding a practical flow situation. It does offer the possibility of yielding (non-physical) check cases for less exact methods. However, comparisons with Garabedian's results as they stand would be difficult, due to their complicated form; he finds W and dW/dz in terms of a parameter t, but does not actually work out z or the complex potential in terms of either t or W.

2.5 Numerical Methods

As is clear from the survey presented thus far, no practical analytical means for handling the nonlinearities in the free-surface boundary conditions has yet been developed. Since the errors of the various approximation schemes are virtually impossible to estimate a priori, there has long been a good deal of interest in purely numerical solutions. Indeed, Wagner (1932) mentions in his fundamental paper that he carried out an iterative solution for the entry of a wedge of 18° deadrise angle, but, regrettably, he gives no details of the solution.

2.5.1 Conical-Flow Problems

Most numerical analyses have, like Wagner's, been restricted to the concial-flow problems of the constant-speed entry of wedges or cones. The great attraction of such problems is that they contain no characteristic length other than the depth of submergence. Hence a solution put in terms of the similarity variables x/h and r/h is otherwise independent of time. The consequences of this fact are far more profound than would appear at first sight. To demonstrate this, we shall follow the derivations of Shiffman & Spencer (1951), after noting that the main results were found by Wagner (1932).

Let s be a Lagrangian coordinate for particles on the free surface; specifically, let s be the distance between the particle and the vertex of the entering wedge or cone when $h = 0$, the time at which the body first hits the water. Further, let $\underline{P}(s,h)$ be the vector from the vertex to the particle labelled with s when $h > 0$. Clearly, from the similarity, there exists a vector function $\underline{\tilde{P}}$ such that

$$\underline{P}(s,h) = h\,\underline{\tilde{P}}(s/h) \tag{65}$$

From this it follows that

$$\begin{aligned} \frac{\partial \underline{P}}{\partial h} &= \underline{\tilde{P}} - \frac{s}{h}\underline{\tilde{P}}' \\ \frac{\partial^2 \underline{P}}{\partial h^2} &= \frac{s^2}{h^3}\underline{\tilde{P}}'' \\ \frac{\partial \underline{P}}{\partial s} &= \underline{\tilde{P}}' \end{aligned} \tag{66}$$

in which the primes denote differentiation with respect to argument.

Now $U^2\partial^2\underline{P}/\partial h^2$ is the instantaneous acceleration of the fluid particle s , while $\partial\underline{P}/\partial s$ is directed tangential to the free surface. Since the pressure on the surface is constant, the pressure gradient at the surface — which by Newton's law is parallel to $\partial^2\underline{P}/\partial h^2$ — is normal to the surface. Thus

$$\partial^2\underline{P}/\partial h^2 \cdot \partial\underline{P}/\partial s = 0$$

or, from (66)

$$\underline{\tilde{P}} \cdot \underline{\tilde{P}}'' = \frac{1}{2}\, d(\underline{\tilde{P}}')^2/d(s/h) = 0 \qquad (67)$$

Thus $|\underline{\tilde{P}}'|$ is a constant. Specifically, considering its behavior as $s/h \to \infty$, and again using (66),

$$|\underline{\tilde{P}}'(s/h)| = 1 = |\partial\underline{P}/\partial s| \qquad (68)$$

Equation (68) shows that the Lagrangian coordinate s is simply the arc length along the free surface, measured from its intersection with the body. Conversely, the arc length is a Lagrangian coordinate, and so the distance along the surface between any two fluid particles is constant in time. Now the free-surface particle in contact with the body at any given time must move parallel to the body surface, and so always was and always will be in contact with the body. Therefore, the arc length from a given fluid particle on the free surface to the contact point equals the distance from the point at which the body hits the surface to the position the particle had at the initial moment of impact. In numerical solution, this fact is often used to check the free-surface shape, as we shall soon see.

Next we note that equations (65) and (66) may be combined into

$$\frac{\partial \underline{P}}{\partial h} = \frac{1}{h}\underline{P} - \frac{s}{h}\frac{\partial \underline{P}}{\partial s} \tag{69}$$

which becomes, on being resolved into our space-fixed coordinates,

$$\begin{aligned} \frac{1}{U}\phi_x &= \frac{\Delta}{h} - \frac{s}{h}\frac{\partial\Delta/\partial r}{\{1+(\partial\Delta/\partial r)^2\}^{\frac{1}{2}}} \\ & \qquad\qquad\qquad\qquad \text{on} \quad x = \Delta(r,h) \\ \frac{1}{U}\phi_r &= \frac{r}{h} - \frac{s}{h}\frac{1}{\{1+(\partial\Delta/\partial r)^2\}^{\frac{1}{2}}} \end{aligned} \tag{70}$$

Now if we take the dot product of (69) with $\partial \underline{P}/\partial s$, and use (68), we find

$$\frac{\partial\Delta}{\partial s} + \frac{1}{U}\frac{\partial\phi}{\partial s} = \frac{1}{2h}\frac{\partial P^2}{\partial s} - \frac{s}{h} \qquad \text{on} \quad x = \Delta(r,h) \tag{71}$$

Since $P^2 = (\Delta+h)^2 + r^2$, and since we can always ignore a constant in the potential, (71) integrates to

$$\phi = \frac{U}{2h}(\Delta^2+r^2-s^2) \qquad \text{on} \quad x = \Delta(r,h) \tag{72}$$

Thus, once the surface shape is known, the fluid velocity and the potential on the surface can be computed directly from (70) and (72). Consequently, numerical analyses of conical-flow problems begin with the selection of a trial free-surface shape, since use of either of these relations yields a reasonable boundary-value problem for the potential.

Let us now consider some specific numerical analyses. In Pierson's (1950) study of wedge entry, the trial free-surface shape is selected so as to satisfy the following criteria:

1. The "continuity condition," that the fluid displaced by the entering body must appear above the undisturbed free surface, follows from the incompressibility of the flow and continuity equation. It may be noted that Wagner's (1932) flat-plate fitting solution, as given by equation (49), does satisfy this condition. For later reference, we call Wagner's result $\Delta_W(r,h)$.

2. The "arc-length condition," that the distance along the free surface from its point of contact with the body to some point at which the surface is essentially undisturbed equals the horizontal distance from the latter point to the axis of motion, is a consequence of the constancy in time of the arc length between any two fluid particles on the surface. Now Wagner's solution does not satisfy this requirement, nor does it exhibit the spray found in experiments (see Fig. 6).

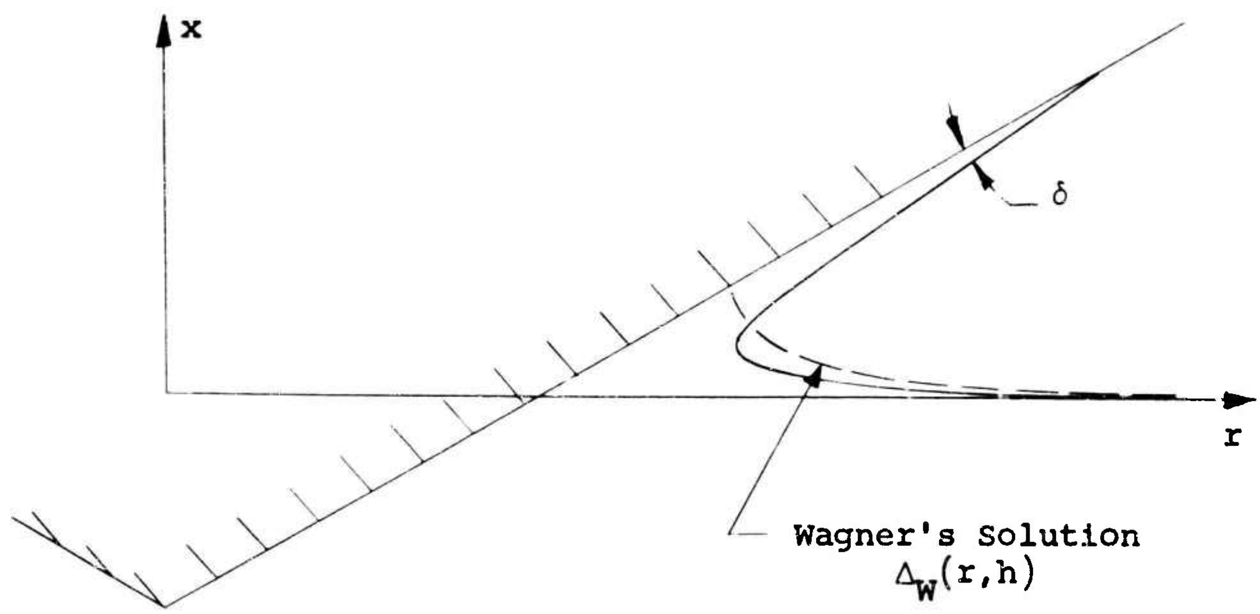


FIGURE 10

PIERSON'S CONSTRUCTION OF A TRIAL FREE-SURFACE SHAPE

Therefore, Pierson "took away" some of the fluid from the Wagnerian shape and used it to form a spray, as follows. Wagner's formula is multiplied by a constant f less than unity to get the shape of the free surface near $x = 0$. The spray portion of the free surface is taken to be triangular in shape, and so is described by the point at which it attaches to the body and by the angle δ formed by the body and the free surface at the attachment point, as shown in Fig. 10. Pierson selected several values of the attachment angle δ, for each of which he "blended" the two parts of the free into one another (for which process he gave no specific recipe), and then adjusted the attachment point and the constant f so as to satisfy the continuity and arc-length requirements.

3. This shape is then further modified, again for each δ, so as to satisfy the "similarity condition". That is, from the shape of the surface, the velocity of the particles thereon may be computed with equation (70). The position of the particles at a later time may then be computed by integration, and the results checked for similarity. Pierson's procedure is actually somewhat more complicated than this, but the idea is the same.

4. Finally, δ is determined by applying what Pierson calls an "irrotationality check". From Green's theorem, if ϕ_1 and ϕ_2 are harmonic in a region bounded by a curve C,

$$\int_C \left[\phi_1 \frac{\partial \phi_2}{\partial n} - \phi_2 \frac{\partial \phi_1}{\partial n} \right] d\ell = 0 \tag{73}$$

where n is the outward directed normal to C. Pierson takes the contour to consist of the right half of the free surface (C_1), the right half of the body (C_2), the axis of symmetry (C_3), and an infinitely large quarter-circle in the fourth quadrant. Now is ϕ_1 is the solution of the entry problem, we may invoke the body boundary condition, symmetry, and the boundary condition at infinity to reduce (73) to

$$0 = \int_{C_1} \left[\phi_1 \phi_{2_n} - \phi_2 \phi_{1_n}\right] d\ell + \int_{C_2} \phi_1 \phi_{2_n} \, d\ell + \int_{C_3} \phi_1 \phi_{2_n} \, d\ell + U \cos\beta \int_{C_2} \phi_2 \, d\ell \tag{74}$$

Since ϕ_1 and its derivatives are known on C_1 from (70) and (72) in terms of the free-surface shape, equation (74) could be reduced to one involving only known quantities and would hence constitute a check on the assumed shape if ϕ_2 could be specified so that its normal derivative vanishes on C_2 and C_3. Clearly, the solution for flow past a wedge could be used for this purpose. Instead, Pierson constructed ϕ_2 by positioning one doublet at the spray root and another at its image with respect to the side of the wedge, so as to make $\partial\phi_2/\partial n$ vanish on C_2, and ignored the contribution of C_3. While this choice emphasizes the shape of the free surface at its spray root, the necessary neglect of the integral over C_3 and the complications which ensue in the evaluation of the integral over C_1 due to the presence of the singularity thereon are serious defects in Pierson's method.

Once the free surface shape is established, the velocity distributions on the wedge are found from a variant of Cauchy's integral formula. Letting $u - iv = dw/dz$, we have

$$\frac{dw}{dz}(z_B) = \frac{1}{\pi i} \int_{C'} \frac{dw}{dZ}(Z) \frac{dZ}{Z - z_B} \tag{75}$$

in which z_B is a point on the wedge, and the contour C' consists of the free surface (on which u and v are known from equations (70)), the wetted surface of the wedge (on which the normal velocity is known from the body boundary condition), and the lower half of a large circle centered at the origin (on which u and v vanish). By rotating the coordinates so that the x-axis (say) is along one side of the wedge, the real part of (75) becomes

$$u(z_B) = -\frac{1}{\pi}\int_{C_2} v(z)\,\frac{dz}{z-z_B} - \frac{1}{\pi}\,\mathrm{Im}\int_{C_2'+C_1} \frac{dw}{dz}(z)\,\frac{dz}{z-z_b} \qquad (76)$$

in which C_2 is the side of the wedge on which the point B is located, C_2' is the other side of the wedge, and C_1 is the free surface. According to Pierson, the contribution of the integral over C_2' is quite small, while the other terms on the right side are known, and straightforward iteration converges rapidly.

Once the velocity distribution along the wedge is known, $\partial\phi/\partial h$ can be computed from the similarity, and so the pressure distribution is readily determined. As a check, the potential was computed at a few points on the wedge from an equation analogous to (75).

While Pierson's analysis contains a number of interesting ideas, the somewhat imprecise method by which the free surface was constructed (in particular, the fairing of the spray into the Wagnerian shape, and the approximation made in the irrotationality check), together with the apparently low accuracy used in the calculation (this was carried out before the dawn of the computer age), prevent us from accepting his results as standards of comparison.

Hillman's (1946) analysis of vertical cone entry also starts by choosing a trial shape of the free surface, and uses integral equations for the value of the potential on the boundary. However, both the free-surface shape and the potential on the cone are prescribed analytically; specifically, they are represented by polynomials. The form of the polynomial for the surface shape is chosen so as to satisfy the asymptotic condition that $\Delta \sim r^{-3}$, which follows from the kinematic boundary condition under the assumption that ϕ is doublet-like far from the body. To determine the coefficients, the arc-length and continuity conditions are imposed on the free-surface shape, along with a requirement (to be discussed below) that the surface be tangent to the cone at the attachment point. The remaining coefficients are determined by satisfying the integral equation which governs the potential at an appropriate number of points. This requires adjustment of the surface shape, so that the potential calculated from the integral equation agrees with that calculated from (72) on the free surface. Since the surface shape enters into the kernel in a complicated fashion (more so than in equation (76), since elliptic functions are involved due to the axial symmetry), Hillman linearized the equations about a trial free-surface shape, the choice of which is not fully motivated, however.

The major advantage of Hillman's procedure over Pierson's is that it can, in principle, be made to yield results of any specified accuracy simply by increasing the orders of the polynomials assumed for the free-surface shape and the potential. Unfortunately, Hillman's requirement that the free surface be tangent to the body, however plausible it may seem at first sight, cannot be justified. Indeed, according to Garabedian (1953), such a situation would imply a singularity at the attachment point. Mackie (1962) has examined the question of the attachment angle in some detail (admittedly with a linearized theory), and finds the angle to be acute but finite both in wedge entry and cone entry. Thus Hillman's results are of uncertain accuracy.

The approach employed by Borg (1957) again begins with choosing a trial free-surface shape which satisfies the arc-length and continuity requirements. The potential on the free surface is calculated from (72), and the potential elsewhere is found by relaxation. The fluid velocities on the assumed free surface are then calculated, and compared with those called for by (70). The free-surface shape is then "altered" (how this is done is not stated) and the procedure repeated until a satisfactory check is obtained. Then the relaxation net is made finer and the process restarted.

Borg worked out only one case, the unsymmetrical entry of a wedge of 90° included angle. Since, as we have noted, other numerical analyses contain errors which make their accuracy suspect, and since these conical-flow problems are the simplest to analyze, there still exists a need for numerical results of this type with which the various approximate methods could be compared. These days any attempt in this direction should certainly take advantage of the existence of high-speed computing machinery. Thus a corrected version of Hillman's approach, which would seem to be most adaptable to programming, is recommended for future work.

2.5.2 Entry of Bodies of Arbitrary Shape

The surface-crossing of non-conical bodies is, of course, considerably more difficult to treat, since the time variation cannot be accounted for by a priori similarity arguments. Recently, an ambitious effort was made at Southwest Research Institute (Chu & Abramson 1959, Chu 1960, Chu & Falconer 1963) to develop a computer program capable of describing the hydrodynamics of the water entry of bodies of arbitrary shape. Of particular interest were blunt two-dimensional shapes such as are typical of ship sections. The free-surface shape was calculated by using a finite-difference version of the kinematic free-surface boundary condition. With the potential on $x = \Delta$ found by integration of the dynamical boundary condition, relaxation methods were used to calculate the potential elsewhere. To start the calculations, the linearized solution of § 2.1 was used.

The method required an impractically long time even on an extremely fast computer. This results from the large number of mesh points required; indeed, machine capacity prevented prescription of as many points in the spray region as would have been liked. But even disregarding this difficulty, the method is rendered inherently inaccurate because of a singularity at the moment of contact.

This singularity was first noted by von Kármán (1929). Its presence is easily deduced from equation (12) and (34), according to which the force per unit length on a cylinder entering water is

$$F_x(h) = \pi\rho U^2 \; R(h) \; R'(h) \tag{77}$$

so that the impact pressure is

$$\lim_{h\to 0} \; F_x(h)/R(h) = \pi\rho U^2 R'(0) \tag{78}$$

which is infinite for blunt bodies.

Note that the singularity is not a consequence of the linearization which underlies the above results, however. Immediately before impact, the potential is constant — zero, according to the initial condition (11) — all throughout the water.

Immediately after impact, the body boundary condition (8) demands that the potential have a finite gradient in the vertical direction (if the entering body is blunt) at the impact point. Since, according to the maximum-modulus principle, the potential must achieve its extreme values on the boundaries of the flow region, this means that the potential at the impact point must differ from its value at infinity, which, in turn, is the same as before impact. Thus the potential at the impact point undergoes a discontinuous change at impact, so that $\partial\phi/\partial t$, and hence p, are infinite at impact.

The difficulties caused by this singularity prevent the results of Chu & Falconer from being of practical value. Nevertheless, their report is valuable for a detailed discussion of the application of relaxation methods to the problem.

2.6 Comparison and Evaluation of Theories

Calculations which cover the entire surface-crossing period, from complete submergence to complete emergence, are relatively few in number. Results for the time history of the net upward force are shown in Figs. 11-13 for the sphere (Shiffman & Spencer 1945a), the slender biconvex airfoil (Moran 1961), and the slender parabolic spindle (Moran 1961), respectively. In all these calculations the free-surface boundary conditions were linearized, and no corrections of the type discussed in § 2.3 were made. Moran also linearized the body boundary condition.

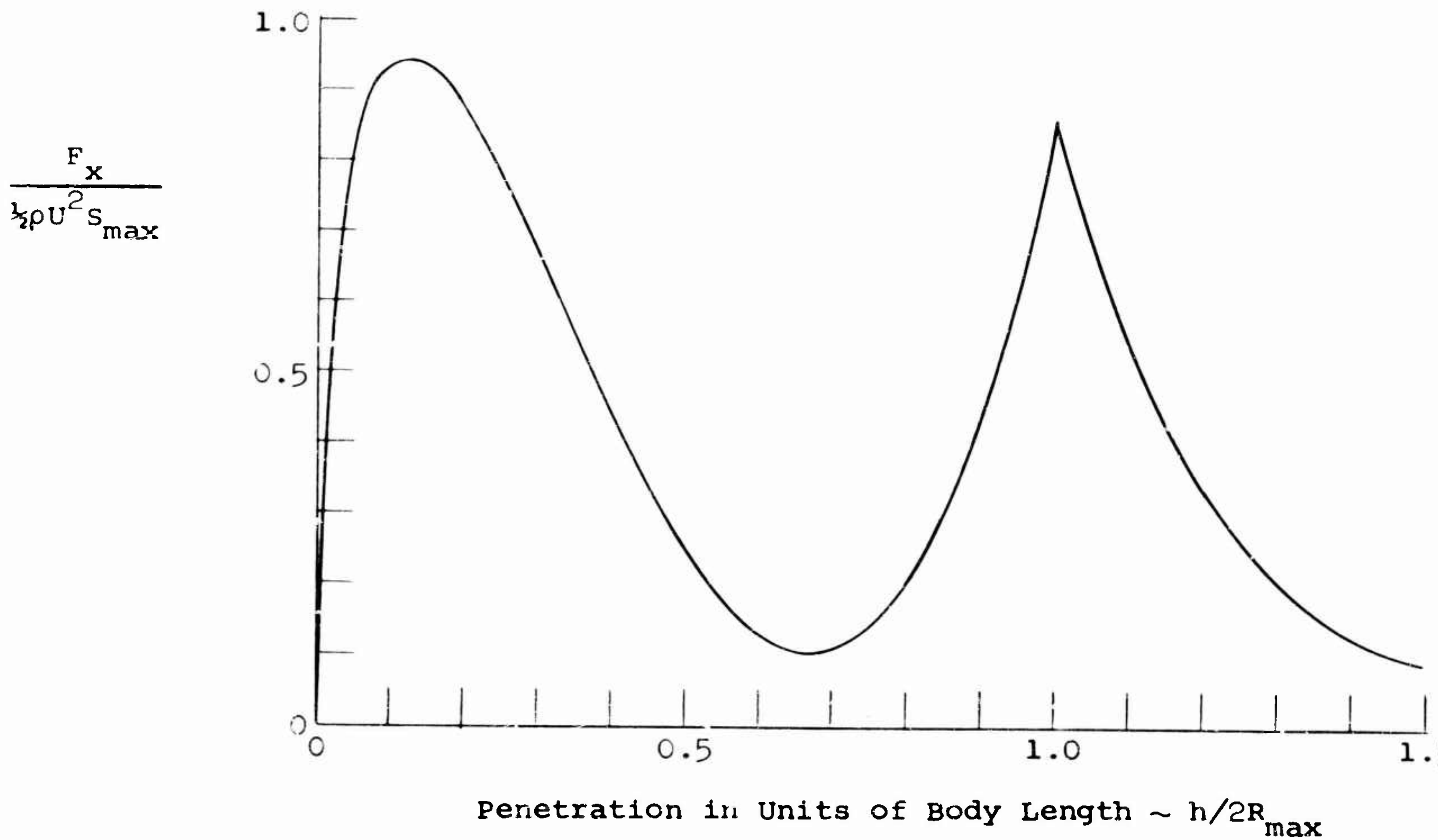


FIGURE 11

UPWARD FORCE ON SPHERE CROSSING WATER SURFACE VERTICALLY
(Shiffman & Spencer 1945a, 1947)

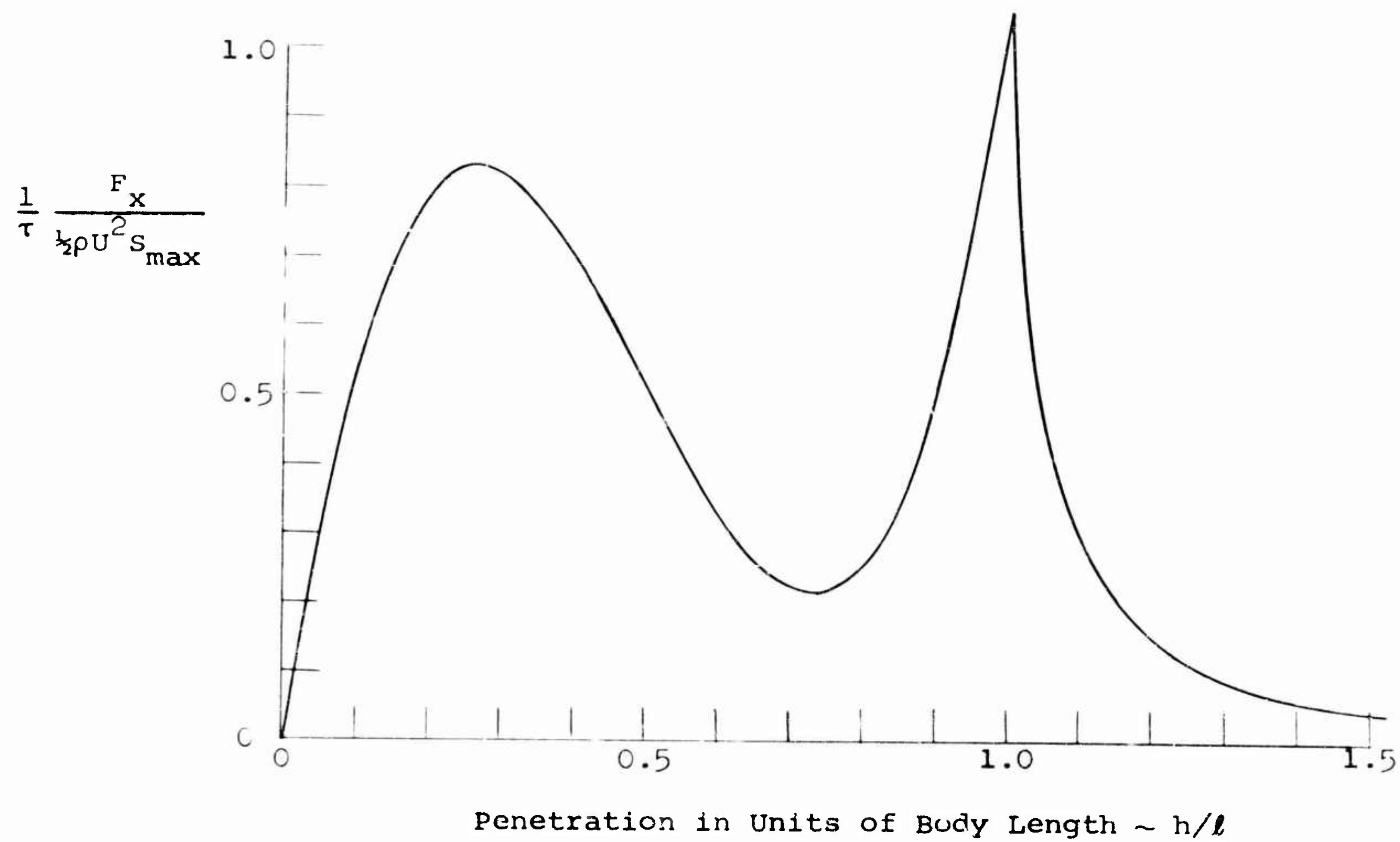


FIGURE 12

UPWARD FORCE ON BICONVEX AIRFOIL CROSSING WATER SURFACE VERTICALLY
(Moran 1961)

$$R = \tau x^*(\ell - x^*)$$

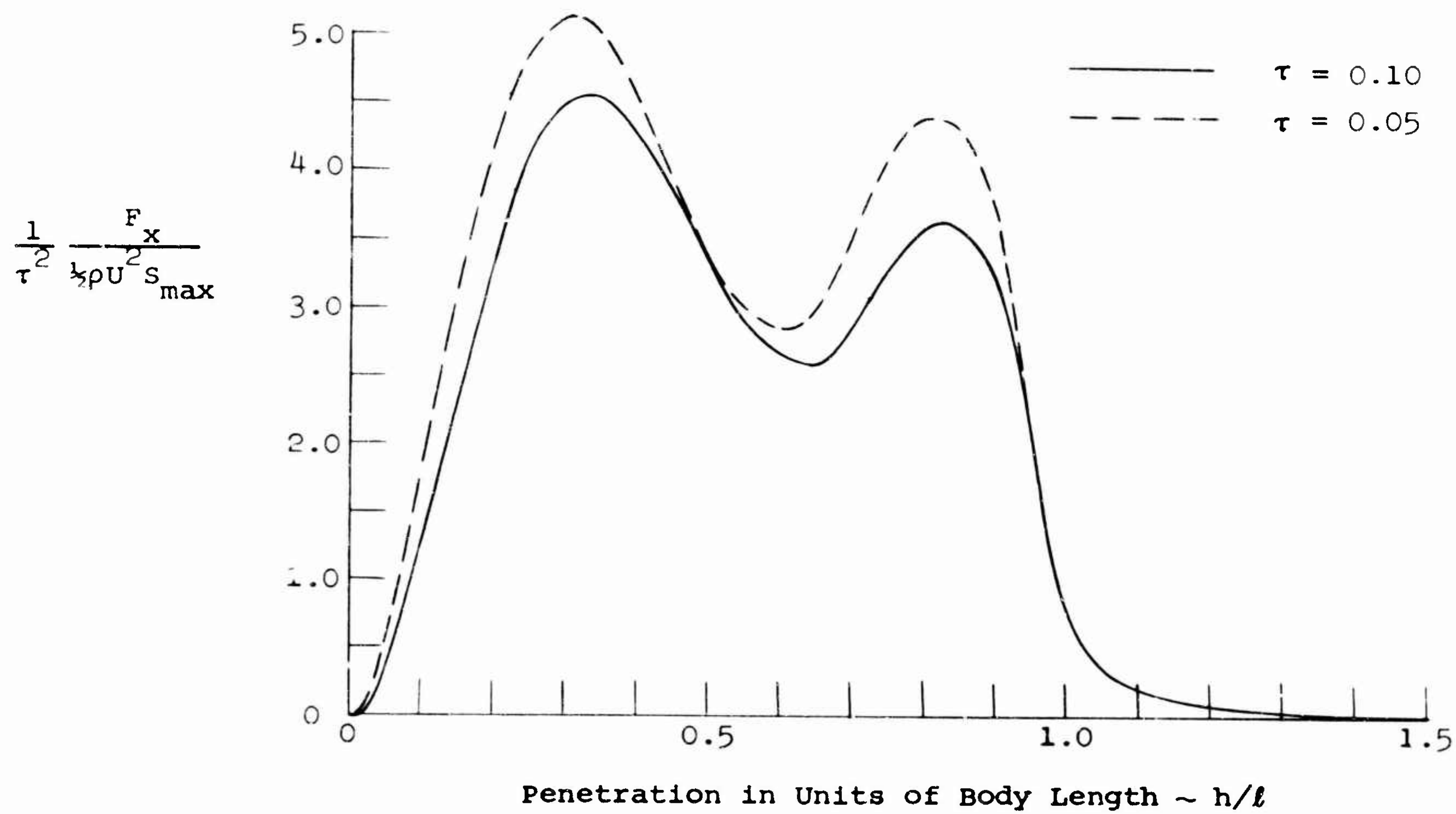


FIGURE 13

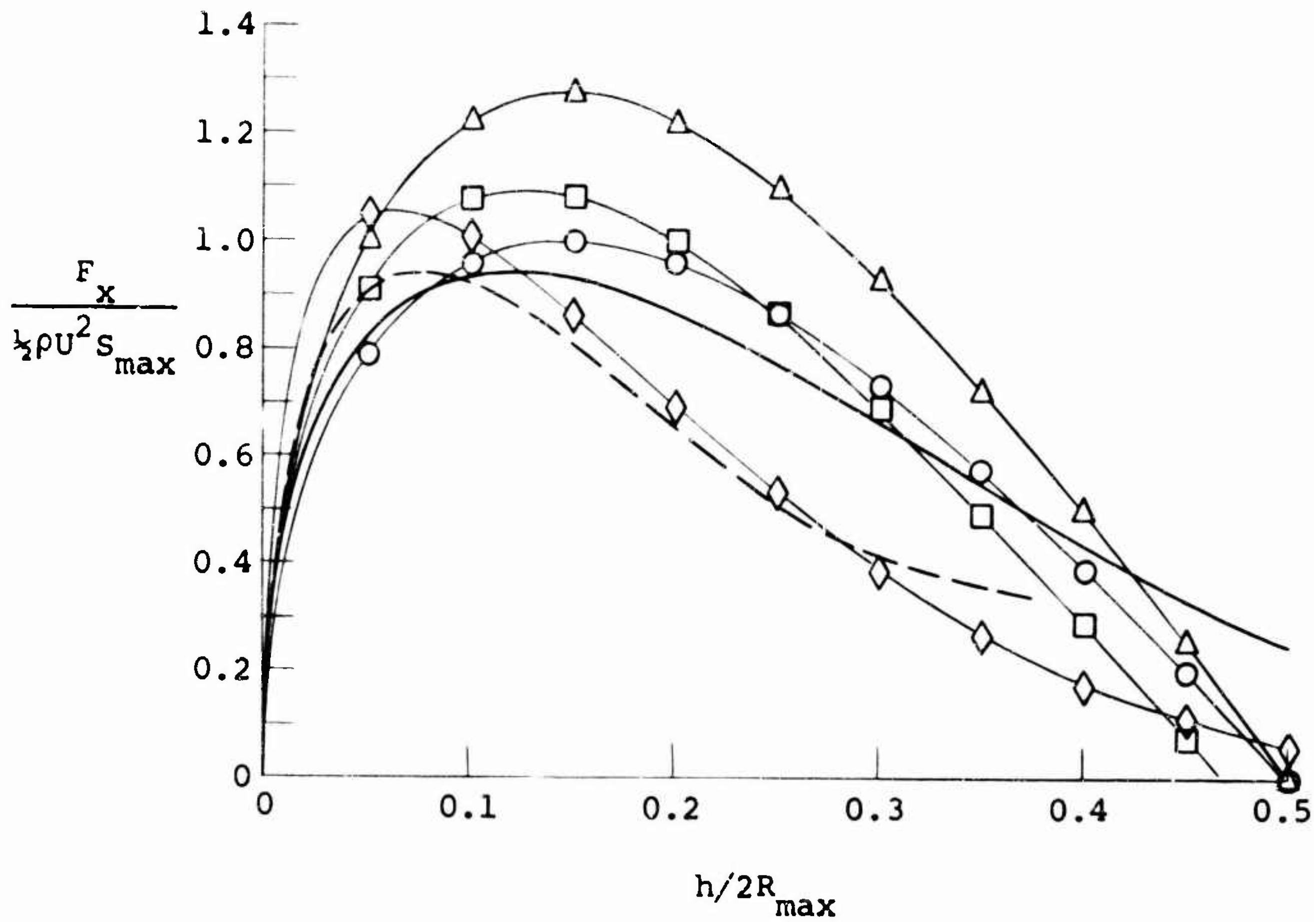
UPWARD FORCE ON PARABOLIC SPINDLE CROSSING WATER SURFACE VERTICALLY
(Moran 1961)

$$R = \tau x^*(\ell - x^*)$$

That the force is generally upward can be explained as follows. Consider, for example, the exit problem. Initially, the fluid has a net momentum upward (equal to the product of the added mass and the body's speed). As the body exits, this momentum must vanish. Thus it is transferred to the body as an upward thrust. Similar reasoning applies to the entry situation.

A good deal of attention has been given to the impact phase of water-entry problems. Figs. 14 and 15 compare the predictions of several theories for the time history of the impact force felt in the early stages of the entry of a sphere and of a circular cylinder, respectively. The variations with deadrise angle of the added masses of cones and wedges are shown in Figs. 16 and 17, respectively, while Fig. 18 compares some predictions as to the pressure distribution on an entering wedge. In these figures, the various curves are distinguished as to the approximating bodies used to calculate the added mass, the wetting correction, and the free-surface correction. The curves derived with Chou's (1946) theory, which cannot be described in these terms, are called "Chou's exact method" if the free surface was regarded as an isobar and "Chou's approximate method" if it were treated as an equipotential. The points calculated by Hillman (1946) and by Pierson (1950) are given the designation "numerical".

As we have indicated in our discussion (§§ 2.1-2.5) of the various analyses which underly the results shown in Figs. 11-18, all of these analyses have some mathematical defect, and



——— Lens fitting (exact linearized solution) (Shiffman & Spencer 1945a)

—△— Disc fitting (Plesset 1942)

—□— Ellipsoid fitting (Shiffman & Spencer 1945a)

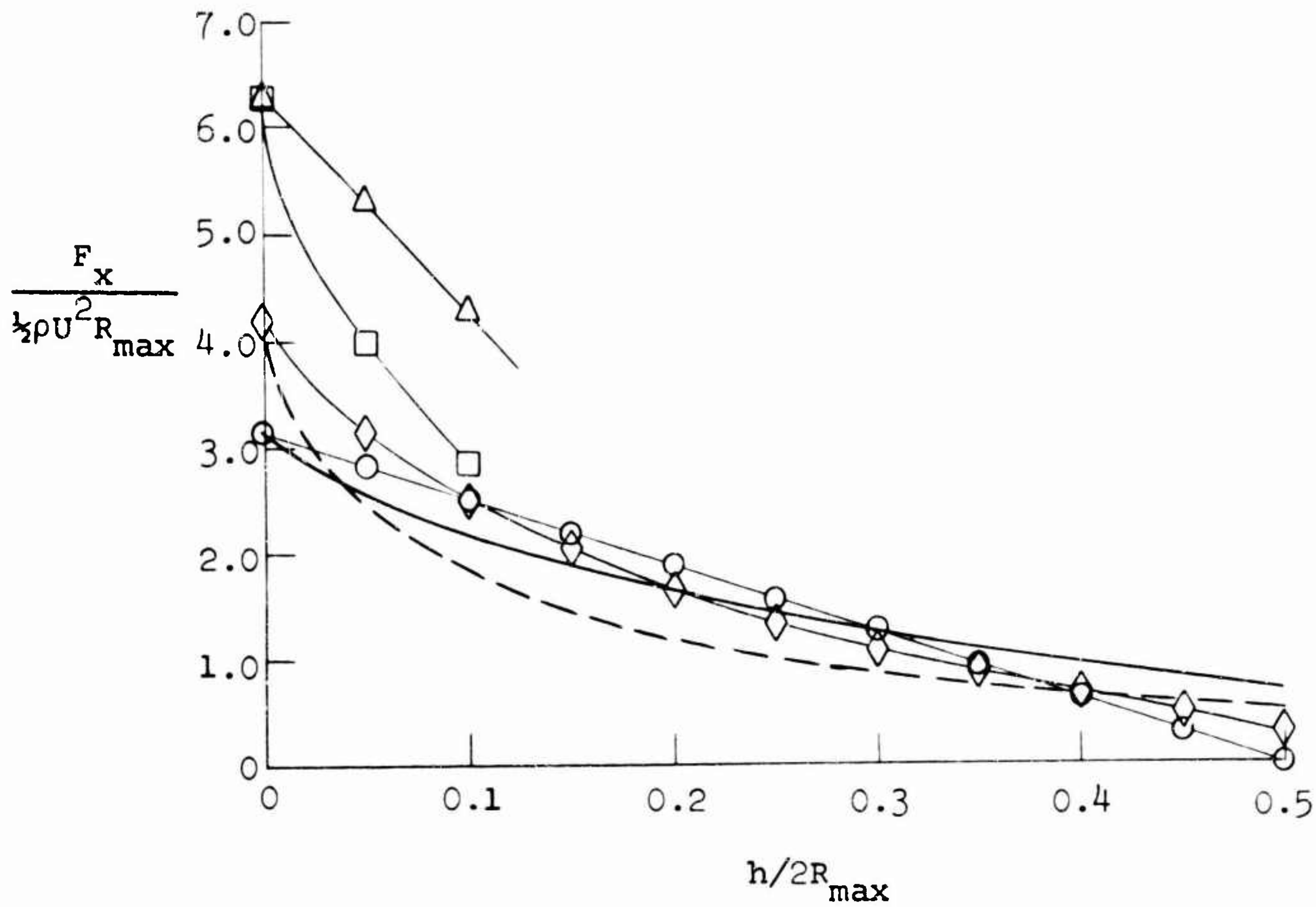
—○— Sphere fitting (Shiffman & Spencer 1945a)

—◇— Lens fitting, with wetting correction by disc fitting + experiment, and free-surface correction by ellipsoid fitting (Shiffman & Spencer 1945b)

– – – – Chou's exact method (Chou 1946)

FIGURE 14

COMPARISON OF THEORETICAL RESULTS FOR IMPACT FORCE ON A SPHERE



Lens fitting (exact linearized solution) (Fabula 1955)

—○— Flat-plate or ellipse fitting (Fabula 1955)

—□— Lens fitting, with wetting correction by ellipse fitting (Fabula 1957)

—△— Flat-plate fitting with wetting correction (Fabula 1957)

– – – – Chou's exact method (Fabula & Ruggles 1955)

—◇— Chou's approximate method (Fabula & Ruggles 1955)

FIGURE 15

COMPARISON OF THEORETICAL RESULTS FOR IMPACT FORCE ON A CIRCULAR CYLINDER

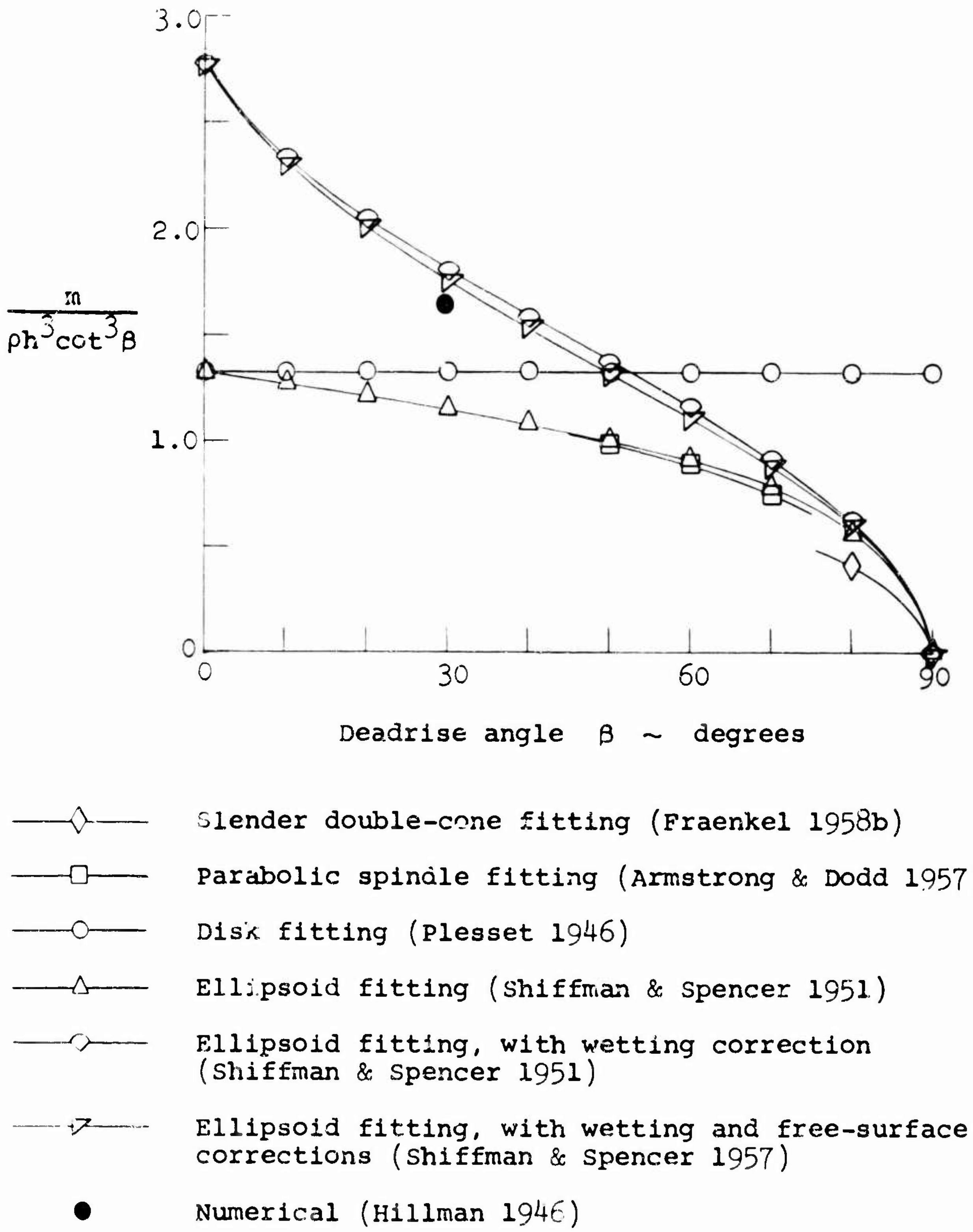
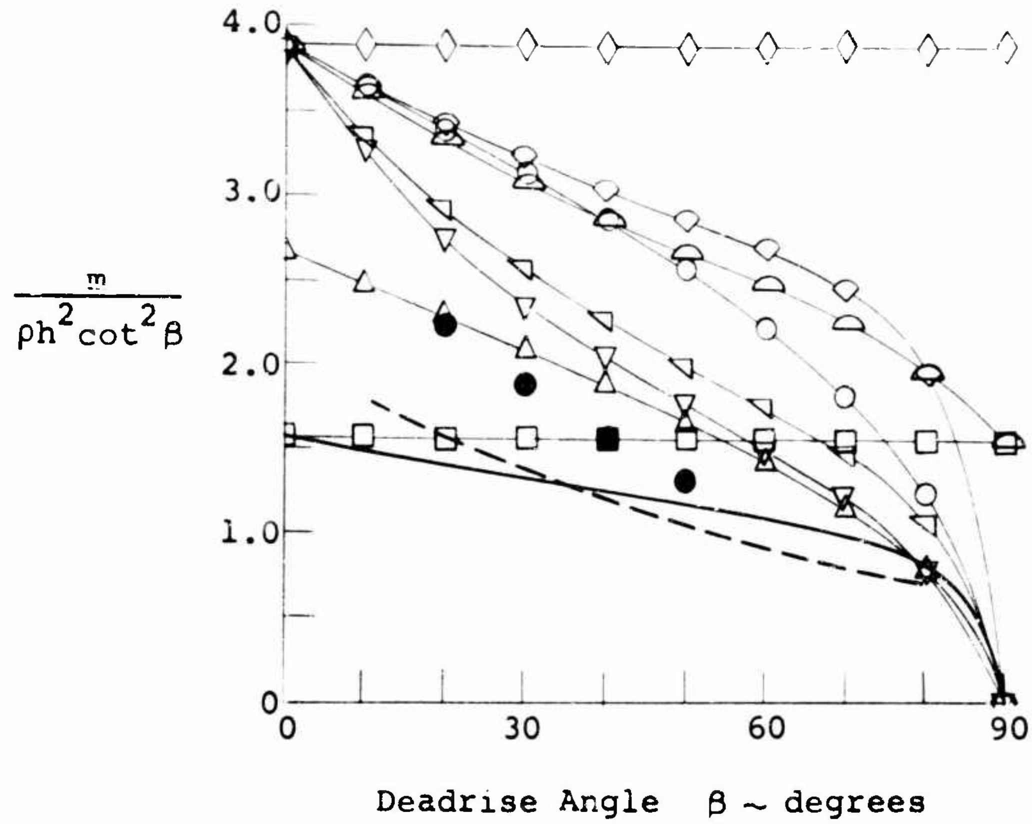


FIGURE 16

COMPARISON OF THEORETICAL RESULTS FOR
ADDED MASS OF A CONE



Diamond fitting (exact linearized solution) (Wagner 1932)

Diamond fitting, with wetting correction (Fabula 1957)

Diamond fitting, with wetting and free-surface corrections (Fabula 1957)

Ellipse or flat-plate fitting (von Kármán 1929)

Ellipse fitting, with wetting correction (Karzas 1952)

Flat plate fitting, with wetting correction (Wagner 1932)

Diamond fitting, with wetting correction by flat-plate fitting (Fabula 1957)

Diamond fitting, with wetting correction by ellipse fitting (Fabula 1957)

Diamond fitting, with wetting and free-surface corrections by ellipse fitting (Fabula 1957)

Chou's exact method (§ 2.4.1)

Numerical (Pierson 1950)

FIGURE 17

COMPARISON OF THEORETICAL RESULTS FOR ADDED MASS OF A WEDGE

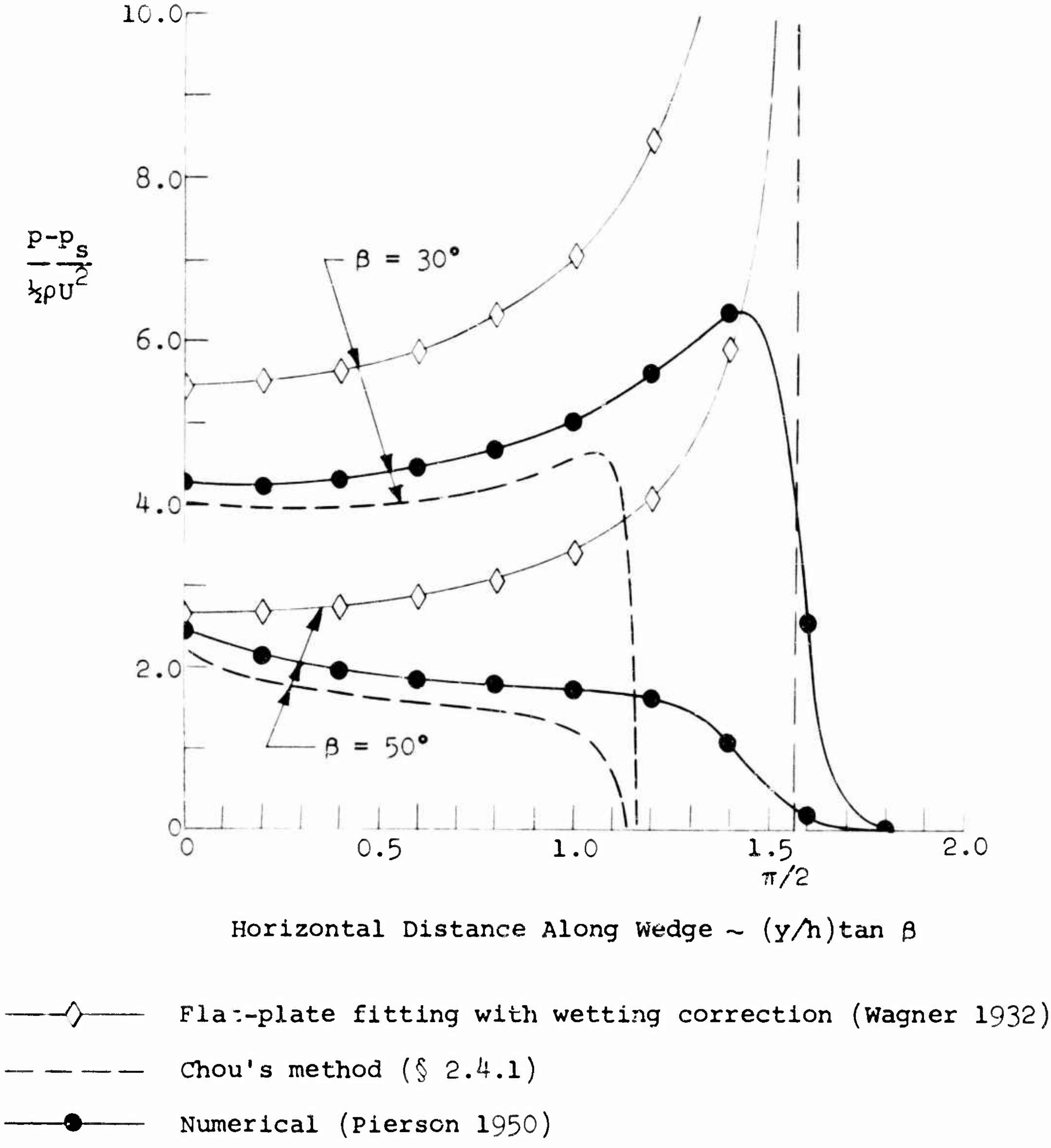


FIGURE 18

COMPARISON OF THEORETICAL RESULTS FOR
PRESSURE DISTRIBUTION ON A WEDGE ENTERING WATER

most are certain to be in serious error during broach, at least as to the details of the flow. Only in the case of complete submergence do the approximations underlying the various theories become justifiable. The extent to which this is true is indicated in Fig. 19, which shows results obtained by Moran & Kerney (1964) for the maximum distortion of the free surface ($\Delta(0,h)$) during the vertical surface-crossing of a slender ellipsoid of revolution according to first- and second-order theory. To make the comparisons refer only to the degree to which the free-surface boundary conditions are satisfied, certain second-order effects having to do only with the body boundary condition are included in the curve marked "First Approximation". It is seen that linearization of the free-surface boundary conditions is unquestionably justifiable when the depth of submergence of the upper end of the body $(h - 1)$ is greater than about 1½ body diameters.

The ultimate test both of the model formulated in § 1.2 and of the various analyses of that model is, of course, in experiments. While accurate data on the loads encountered in surface-crossing are almost as difficult to obtain experimentally as they are from theory, there nevertheless exists a fair amount of reliable data with which the results discussed here may be compared. Data on the impact force felt in water entry are particularly abundant, and are compared with the relevant theoretical predictions by Shiffman & Spencer (1945b), Chou (1946), and Nisewanger (1961) for the case of sphere entry; by Shiffman & Spencer

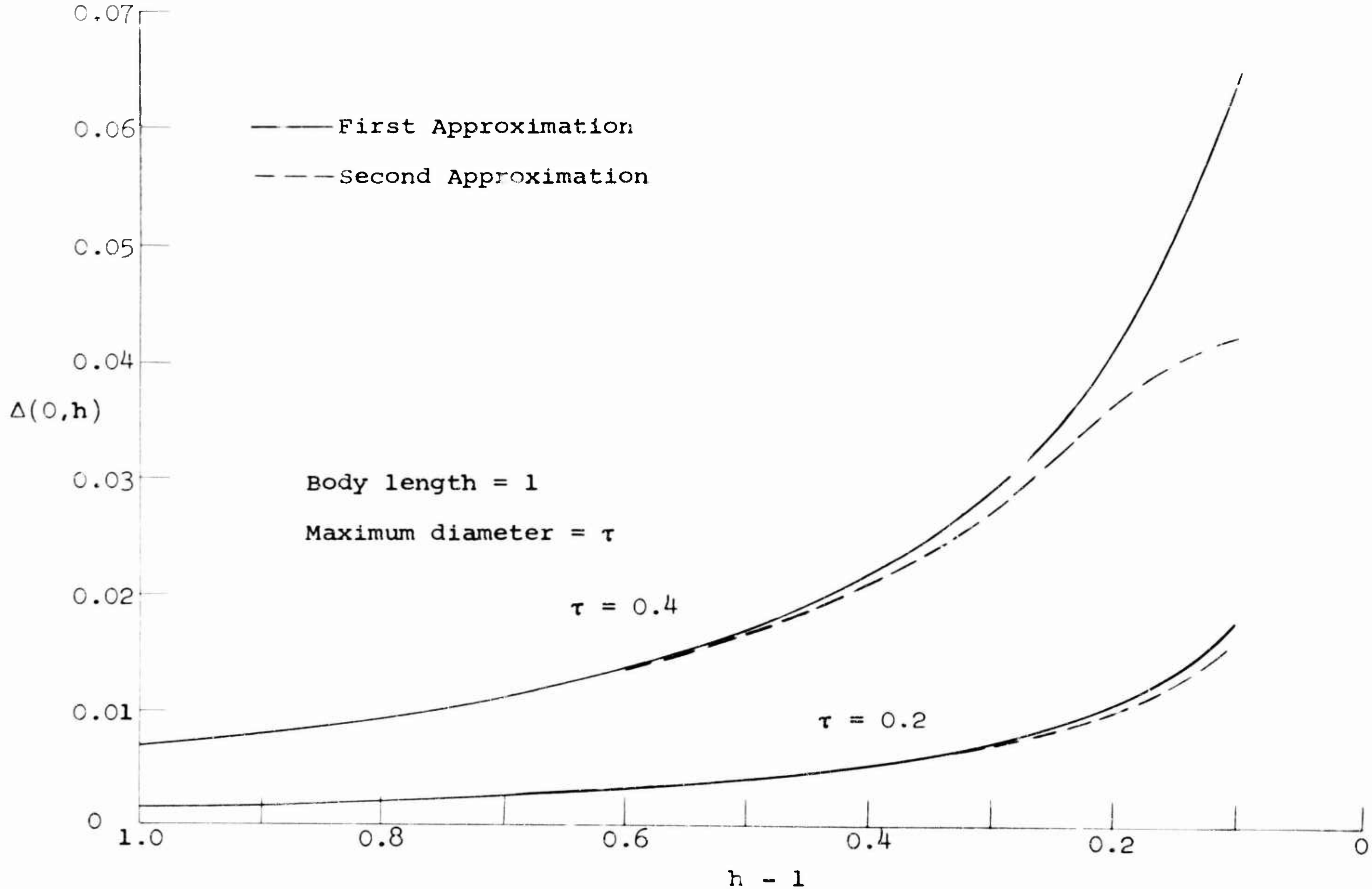


FIGURE 19

MAXIMUM DISTORTION OF FREE SURFACE DUE TO
VERTICAL EXIT OR ENTRY OF SLENDER ELLIPSOID OF REVOLUTION
(Moran & Kerney 1964)

(1951) for cone entry; and by Chu & Abramson (1959) for the two-dimensional entry of wedges and circular cylinders. Clark & Robertson (1960) have attempted to measure the history of the upward force thru the submerged and broach phases of water exit, but their data — obtained by twice differentiating photographic histories of the trajectory, rather than by using the more accurate accelerometer technique — exhibit large scatter.

Measurements of pressure distributions on the body are even more difficult to carry out than are force measurements. This is partly due to the transient nature of the problem. Also, the finite size of the gages, which must of course be fully wetted before they can register, makes it absolutely impossible to measure the pressure in the immediate vicinity of the point at which the free surface intersects the body. Within these limitations, pressure distribution data are available for the entry of the circular cylinder (Schnitzer & Hathaway 1953) and of the sphere (Nisewanger 1961).

Rather than further complicate our curves, we refer the reader to the aforementioned reports for comparisons between theory and experiment. Such comparisons can be summarized as follows. Theory is in fair agreement with experiment for the impact forces felt by spheres and cones. For these cases, it makes little difference whether ellipsoid fitting is used to calculate the added mass or whether the exact linearized problem is solved, but the wetting and free-surface corrections are in the right direction.

Also, Chou's (1946) theory is as good as any other. There is at least qualitative agreement between theory and experiment on the general shape of the force vs. time curve, see Figs. 11-13, and on the occurrence of a peak in the pressure distribution near the edges of the wetted region, see the blunt wedge case of Fig. 18.

However, theory is wrong even as to trends in two-dimensional impact. The added mass coefficient of a wedge, defined as the ordinate of Fig. 17 generally decreases with increasing deadrise angle according to theory, but actually increases for small deadrise angles (Bisplinghoff & Doherty 1952). Also, the initial impact force in the entry of a cylinder is not finite, as predicted in Fig. 16, but is zero (Schnitzer & Hathaway 1953). Especially in the latter case, these erroneous predictions reflect the infinite impact pressure found in the theory of blunt-body entry, which anomaly was noted in § 2.5.2 to be a defect of the formulation of § 1.2 rather than of the analyses based on that formulation. Of course, the impact pressure felt by a blunt-nosed three-dimensional body is also infinite, but the nature of the singularity is such that it shows up as a finite impact force only in the two-dimensional case.

CHAPTER THREE

"UNCONVENTIONAL" SOLUTIONS

As noted in § 1.2, the formulation set forth therein is not universally followed. This partly reflects the sometimes serious discrepancies noted in § 2.6 to exist between results obtained under that formulation and experiment, but often is simply due to the fact that it is possible to analyze some of the factors neglected in the "conventional" formulation. This chapter presents a survey of both types of extensions to the basic solutions discussed in Chapter Two.

Before we begin the discussion, however, we reiterate the remarks made in § 1.2 regarding the neglect of cavitation. This phenomenon certainly has a substantial influence on the loads felt in surface crossing, especially in the latter stages of entry problems, and in high-speed exit. Unfortunately, the analysis of unsteady cavitation even in a two-dimensional flow bounded by a free surface is beyond the scope of the present state of the art. Extensive (and continuing) experimental studies at the Naval Ordnance Laboratory (under the direction of A. May) and the Naval Ordnance Test Station (under J. G. Waugh) have resulted in some beautiful photographs and good hydroballistic data, but have yielded little of aid to the hydrodynamicist. On the other hand, it is quite clear that cavitation is not responsible for all of the serious disagreements between theory and experiment reported in § 2.6, especially those connected with the infinite impact pressure predicted for blunt-body entry. Thus, the ensuing discussion, while incomplete since it ignores cavitation, is certainly not meaningless.

3.1 Effects of Variable Entry Speed

Though the accelerations due to surface crossing can be considerable, they are of such short duration that the total velocity change is usually only a few percent of the initial speed (Breslin & Kaplan 1957). Nevertheless, it is of interest to note that the effects of variable entry speed are easily calculated within the approximations of the linearized theory, as was noted by von Kármán (1929) and Wagner (1932).

The crucial point is that the linearized free-surface boundary conditions can still be written as (25) and (26) even when U is variable. Thus the problems of determining ϕ and Δ are still decoupled, and the potential problem can still be reduced to one of steady unbounded flow, as in Fig. 2. From this quasi-steadiness and the form of the body boundary condition (8), the potential is directly proportional to the instantaneous value of U, but otherwise depends on time only parametrically through the depth of submergence $h(t)$; i.e.,

$$\phi = U(t)\ F(x,r,h(t)) \tag{79}$$

Then, from (13), the apparent mass is a function of h but is otherwise independent of t. In the important case where the body is in free fall during its passage thru the surface, Newton's law and equation (12) yield as a result

$$M \frac{dU}{dt} = - \frac{d}{dt} U\, m(h) \tag{80}$$

where M is the mass of the body. Integrating, we obtain

$$U \equiv \frac{dh}{dt} = \frac{MU_o}{M + m(h)} \tag{81}$$

where U_o is the body speed at $h = 0$ (before entry or after exit). Thus, once the potential problem has been solved parametrically as a function of h, $U(h)$ can be found from (81), $h(t)$ can be found by integrating (81), and hence the pressure on the body can be calculated from (5), the free-surface distortion from (26), etc.

3.2 Effects of Gravity

Gravity has both direct and indirect effects on the body's pressure distribution. As can be seen from equation (5), gravity induces a linear variation of the pressure with depth, which leads on integration to the familiar force of buoyancy, which need not be further discussed here. Of greater interest is the indirect effect of gravity, which arises because of the modification of the dynamical free-surface boundary condition from (7) without gravity to

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + g\Delta = 0 \qquad \text{on} \quad x = \Delta(r,t) \tag{82}$$

with gravity.

Available analyses of the effects of gravity all make the assumption of small disturbances on the free surface, and so linearize the associated boundary conditions. After dropping the quadratic term from (82), applying what's left on $x = 0$, differentiating with respect to time, and substituting for Δ_t from (26), we obtain

$$\phi_{tt} + g\,\phi_x = 0 \qquad \text{on} \quad x = 0 \tag{83}$$

which replaces (25) in the formulation of the linearized problem.

Most analyses of surface-crossing at finite Froude number refer to water-exit situations (as noted in § 1.4 the flow is irreversible when gravity is not negligible). Sakai, Husimi, & Hatoyama (1933) considered vertical variable-speed exit in two dimensions of a circular cylinder, which they represented by a point doublet of strength proportional to the exit speed. Their solution was obtained by a Fourier-transform technique directly analogous with Lamb's (1913) treatment of the motion of a submerged cylinder parallel to the surface. Integration of a linearized pressure formula yielded for the net upward force per unit length

$$\frac{F_x}{\rho U^2 a} = \pi \left[\frac{1}{4d^3} - \frac{F}{2d^2} + \frac{F^2}{d} - 2F^3 e^{2dF} E_1(2dF)\right] \qquad (84)$$

where E_1 is the exponential integral, F is the square of the Froude number,

$$F \equiv U^2/ga \qquad (85)$$

a is the cylinder radius, and d is the depth of submergence of the center of the cylinder, $h - a$.

Corresponding formulas for the exit of a three-dimensional doublet (and hence for the exit of a sphere) or of other singularities may be worked out by specializing the general results for the motion of a point singularity of time-dependent strength along an arbitrary path beneath a free surface, which was derived via integral transform methods by Haskind (1946) and Brard (1948); see also Wehausen & Laitone (1960). In the special case of vertical exit of a constant-strength singularity, the solution can be expressed in terms of image singularities, as was shown by Moran (1964a) through the unusual procedure of (i) assuming an expansion in even powers of the Froude numder, (ii) determining the general term of the expansion, and (iii) showing that the expansion satisfies an ordinary differential equation which, on integration, yields a solution in closed form valid for all Froude numbers. Specifically, the fundamental solution corresponding to vertical constant-speed motion of a source of constant strength was found to consist of the image sink of the infinite-Froude-number solution, plus a continuous distribution of sources along the vertical line extending upward from the image sink to infinity. The strength of the distribution decays exponentially with increasing distance from the image point. Both the strength and the rate of decay are inversely proportional to F.

Aside from these fundamental solutions, analyses of surface crossing at finite Froude number are based on slender-body theory. Moran (1961) assumed an expansion in inverse powers of F to treat the water-exit at large but finite Froude numbers of slender bodies of revolution and symmetric airfoils. More recently, he distributed the fundamental solution described above to treat the case of arbitrary Froude number (Moran 1964b). In the former paper, complete time histories (including the broach phase of motion) of the axial force felt by a biconvex airfoil and by a parabolic spindle were calculated, while the latter paper presents results for the submerged part of the water-exit of a slender ellipsoid of revolution.

The hydrodynamic contribution to the upward force (i.e., buoyancy is neglected) felt by a slender ellipsoid as it approaches the surface from below is plotted in Fig. 20 as a function of time. It is seen that the effect of gravity, outside of buoyancy,

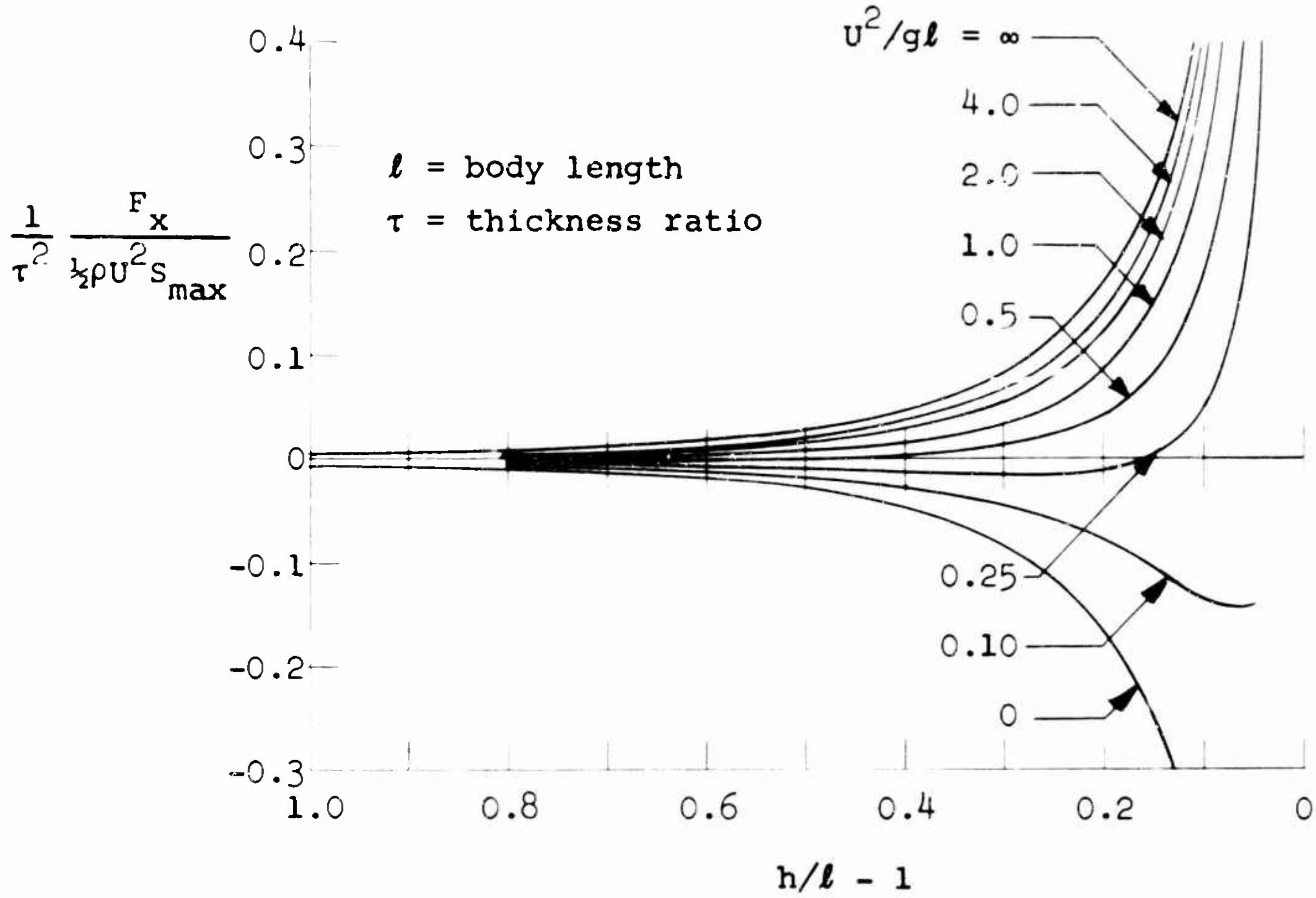


FIGURE 20

EFFECT OF FROUDE NUMBER ON HYDRODYNAMIC PART OF UPWARD FORCE ON SLENDER ELLIPSOID OF REVOLUTION IN VERTICAL WATER EXIT (Moran 1964b) is to reduce the upward force from that felt at infinite Froude number. This reflects the fact that, at finite Froude number, part of the fluid energy is used to set up wave motions rather than being transferred to the body. However, from Fig. 20 and the results obtained thru the broach phase of the motion by Moran (1961), we do not expect these Froude-number effects to be appreciable at practical values of that parameter. This holds even more emphatically for the lateral forces and moments felt in vertical exit, which are independent of Froude number according to slender-body theory (Moran 1964b).

The above remarks refer to water exit in an initially calm sea. In view of the linearization, the effect of waves can be included by superposition. Breslin & Kaplan (1957) so calculated the effects of plane waves on the transverse force and pitching moment acting on a slender exiting body of revolution, and found the effects to be significant if the waves are high enough. The effects of waves on slender bodies in oblique exit have been considered by Nelson (1961) and Cuthbert & Kerr (1962), the latter authors making a non-specific claim that Nelson's work contains several errors. In both analyses, however, the image of the body in the free surface was completely ignored. While, as Goodman (1960) showed (see § 2.2.2), this is justifiable within the limits of the linearized theory for the vertical-exit case, a more careful analysis ought to be made for the oblique-exit situation.

Recently, the effects of gravity during the water entry of slender bodies were considered by Mackie (1963), who used the integral transform technique. Unfortunately, his results are in the form of integrals, and refer mainly to the free-surface shape. However, reasoning as in the exit case discussed above, we would expect the hydrodynamic contribution to the decelerating force felt in entry to be greater than that felt at infinite Froude number, because of the wave motions set up during entry, but we would not expect the increase to be very large.

3.3 Effects of Water Compressibility

The effects of variable entry speed and of gravity, while interesting and sometimes important, clearly have nothing to do with the most striking failure of the conventional formulation, viz., its demand that the impact pressure felt by a blunt-nosed body be infinite. In his pioneering paper, von Kármán (1929) ascribed this defect to the neglect of the slight compressibility of the water. He reasoned that, when a flat body of surface area A strikes the surface, the ensuing disturbance travels at the speed of sound in the water, c (say). Thus the mass of fluid accelerated in time δt is $\rho A c\, \delta t$. Since the velocity of this mass is increased from 0 to U in time δt, the force acting on the fluid is, by Newton's law, $(\rho A c\, \delta t)(U/\delta t)$ and so the pressure felt by the body is $\rho c U$. In the incompressible-flow approximation, $c \to \infty$. Thus, the reasoning goes, the singularity at impact predicted by conventional theory can and ought to be removed by accounting for the compressibility of the water.

Von Kármán's analysis clearly applies only at impact. The first attempt to describe compressibility effects at later times was made by Trilling (1950a), whose formulation has been used by practically all subsequent investigators. Basic to the analysis is the assumption that the fluid velocity is everywhere small compared to the sound speed. Neglecting viscosity, it then follows that the flow is irrotational, and so can be derived from a velocity potential. The continuity equation is linearized by the small-Mach-number assumption as follows:

$$\nabla \cdot \underline{q} = \nabla^2 \phi = -\frac{1}{\rho}\frac{D\rho}{Dt} \simeq -\frac{1}{\rho c^2}\frac{Dp}{Dt} \tag{86}$$

where the undisturbed values of ρ and c are taken. Bernoulli's equation is also linearized, and so we have

$$p \approx -\rho\, \phi_t \tag{87}$$

exactly as in the linearized incompressible case (with gravity neglected). Combining (86) and (87), we find that ϕ satisfies the wave equation,

$$\nabla^2 \phi = \frac{1}{c^2}\phi_{tt} \tag{88}$$

Since compressibility effects are expected to be important primarily during the initial stages of water entry, the body boundary conditions are applied on the undisturbed free surface in the form

$$\phi_x = -U \quad \text{on} \quad x = 0 , \quad r < R(h) \tag{89}$$

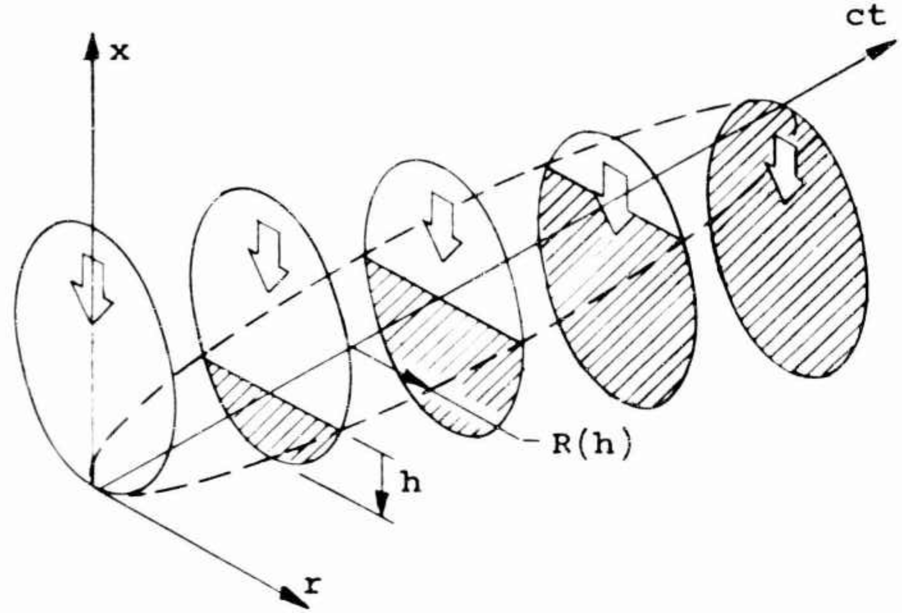
where $R(h)$ locates the intersection of the body with the undisturbed free surface. That is, von Kármán's (1929) flat-plate fitting technique is used. The free-surface boundary conditions become, on linearization, the same as in the incompressible case, viz.,

$$\phi = 0 \qquad \text{on} \quad x = 0 , \ r > R(h) \tag{90}$$

$$\Delta_t = \phi_x \qquad \text{on} \quad x = 0 , \ r > R(h) \tag{91}$$

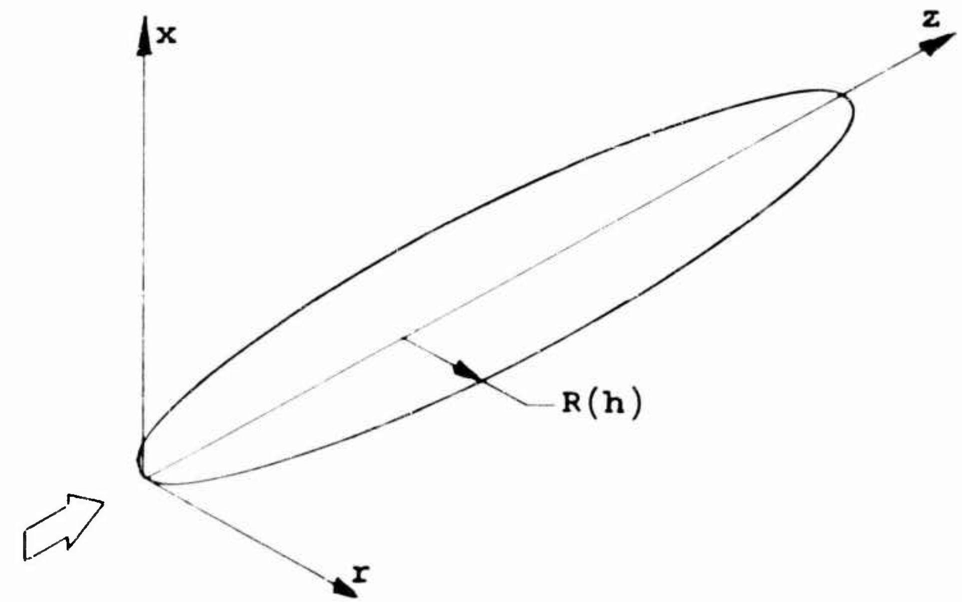
In the two-dimensional case, Trilling (1950a) noted that the boundary-value problem posed by equations (88)-(90) is identical with the problem of determining the lift of a flat wing in supersonic flow. The correspondence is detailed graphically in Fig. 21. Note, in particular, that the time coordinate in the entry problem is equivalent to the streamwise coordinate in the wing problem.

Thus, solutions of the entry problem can be taken directly from the wing-theory literature, extensive references to which are contained in the books by Ward (1955) and by Jones & Cohen (1960).



Water-Entry Problem

Submerged Portion of Cross Section
Indicated by Shading



Equivalent Wing-Theory Problem

FIGURE 21

RELATION BETWEEN LINEARIZED PROBLEMS
OF TWO-DIMENSIONAL ENTRY INTO COMPRESSIBLE WATER AND
OF SUPERSONIC FLOW PAST A THREE-DIMENSIONAL WING

Triliing used this equivalence to find the pressure distribution on the face of a rectangular cylinder for $t < R/c$, at which time the "Mach cones" from the two edges of the equivalent rectangular wing intersect. Ogilvie (1962) gives full details of the solution up to $t = 2R/c$, at which time those Mach cones have crossed each other and have hit the opposite side edges. The pressure distribution on the face of the rectangle is found to be as follows:

$$\begin{aligned}
\frac{p}{\rho c U} &= 1 && \text{for } 0 < |r| < R - ct < R\,, \\
&= \frac{1}{2} - \frac{1}{\pi}\sin^{-1}\left[2\,\frac{|r|-R}{ct} + 1\right] && \text{for } 0 < |R - ct| < |r| < R \\
&= \frac{1}{\pi}\sin^{-1}\left[2\,\frac{|r|+R}{ct} - 1\right] - \frac{1}{\pi}\sin^{-1}\left[2\,\frac{|r|-R}{ct} + 1\right] && \\
& && \text{for } 0 < |r| < ct - R < R
\end{aligned} \tag{92}$$

These formulas are plotted in Fig. 22. Note that $p = 0$ for $ct = 2R$ all along the face of the rectangle. By integration of equations (92), the upward force on the body is found to be

$$F_x = 2\rho c U R\left[1 - \frac{ct}{2R}\right] \qquad \text{for } ct \leq 2R \tag{93}$$

Calculations of the equivalent wing problem in the region corresponding to $t > 2R/c$ are quite difficult, but have been carried out for $t \leq 4R/c$ by Gunn (1947) using Laplace transforms with

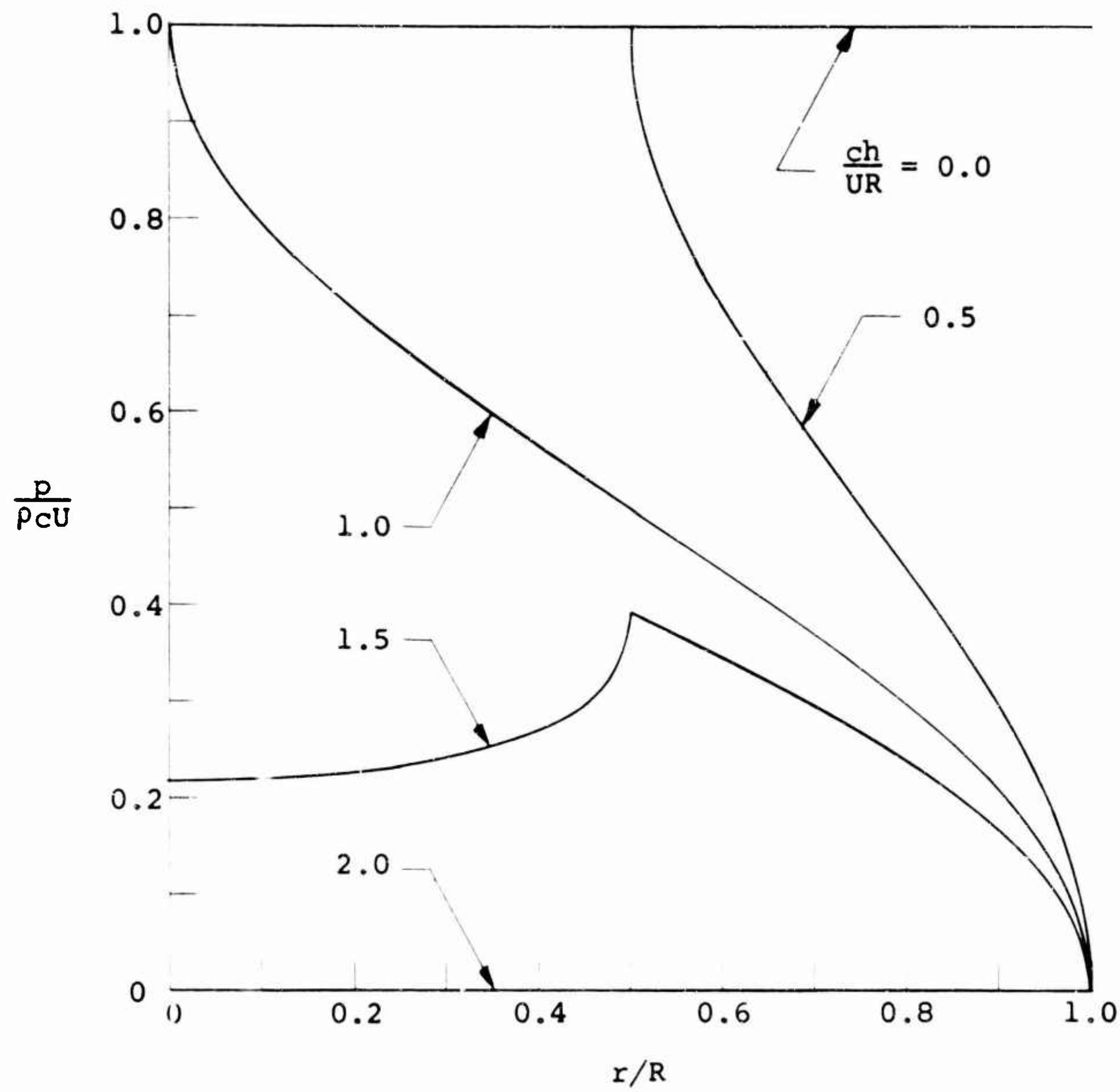


FIGURE 23

PRESSURE DISTRIBUTION ON RECTANGULAR CYLINDER
ENTERING COMPRESSIBLE WATER
(Ogilvie 1962)

respect to the streamwise variable, by Lagerstrom & Graham (1947) using conical-field superposition, and by Behrbohm (1952) using source superposition. Stewartson (1950) used the Laplace transform method to find the behavior of the solution asymptotically far downstream of the leading edge. Applying these results to the present problem, we find that the pressure is negative over the entire face of the rectangle for $2R < ct < 4R$ and for $t \to \infty$, and hence that the net force is then downward (!).

As $c \to \infty$, we expect to recover the incompressible-flow limit. Since the solution is evidently a function of ct, we therefore require the solution for a lifting slender semi-infinite rectangular wing far downstream of the leading edge. As noted above, this has been given by Stewartson. But clearly, in the limit as $ct \to \infty$, the equivalent problem becomes the flow past a lifting doubly infinite slab, which, in the cross-flow plane, is the same as two-dimensional flow past a line segment. Thus the incompressible-flow limit is von Kármán's (1929) solution, as would be expected.

Egorov (1956) has also considered entry of a rectangular cylinder, but assumes early in the analysis that the time dependence of the potential is completely contained in an exponential factor $e^{-\sigma t}$.* After some complicated expansions involving

*Note that this suggests the Laplace-transform method used by Gunn and Stewartson.

Mathieu functions are worked out on this premise, the force is obtained as a function of time, and Newton's law is invoked to determine the parameter σ . Though the analysis seems to work out, there is, in fact, no justification for the assumed time dependence, and the results cannot be correct.

Besides treating the rectangle problem, Trilling (1950a) gave the pressure distribution of a slender wedge (with included angle small compared to the "Mach angle" $\tan^{-1}\frac{c}{U}$), which he showed to reduce to that predicted by von Kármán's theory as $U/c \to 0$. The wedge-entry problem has recently been reconsidered by Skalak & Feit (1963), who give full details of the solution for all values of the "edge Mach number" $(U/c)\ \mathrm{ctn}\ \beta$, which is defined as the ratio to the sound speed of the lateral speed of the intersection of the body with the undisturbed free surface. In these cases the associated supersonic flow problem is, of course, the flat delta-wing at angle of attack. Pressure distributions are plotted in Fig. 23. The upward force is found to be

$$\begin{aligned} \frac{F_x}{\rho c U h\ \mathrm{ctn}\ \beta} &= 2 && \text{for} \quad U\ \mathrm{ctn}\ \beta > c \\ &= \pi \frac{U\ \mathrm{ctn}\,\beta}{c\ E(k)} && \text{for} \quad U\ \mathrm{ctn}\ \beta < c \end{aligned} \tag{94}$$

where, in the second formula, the modulus k of the complete elliptic integral is

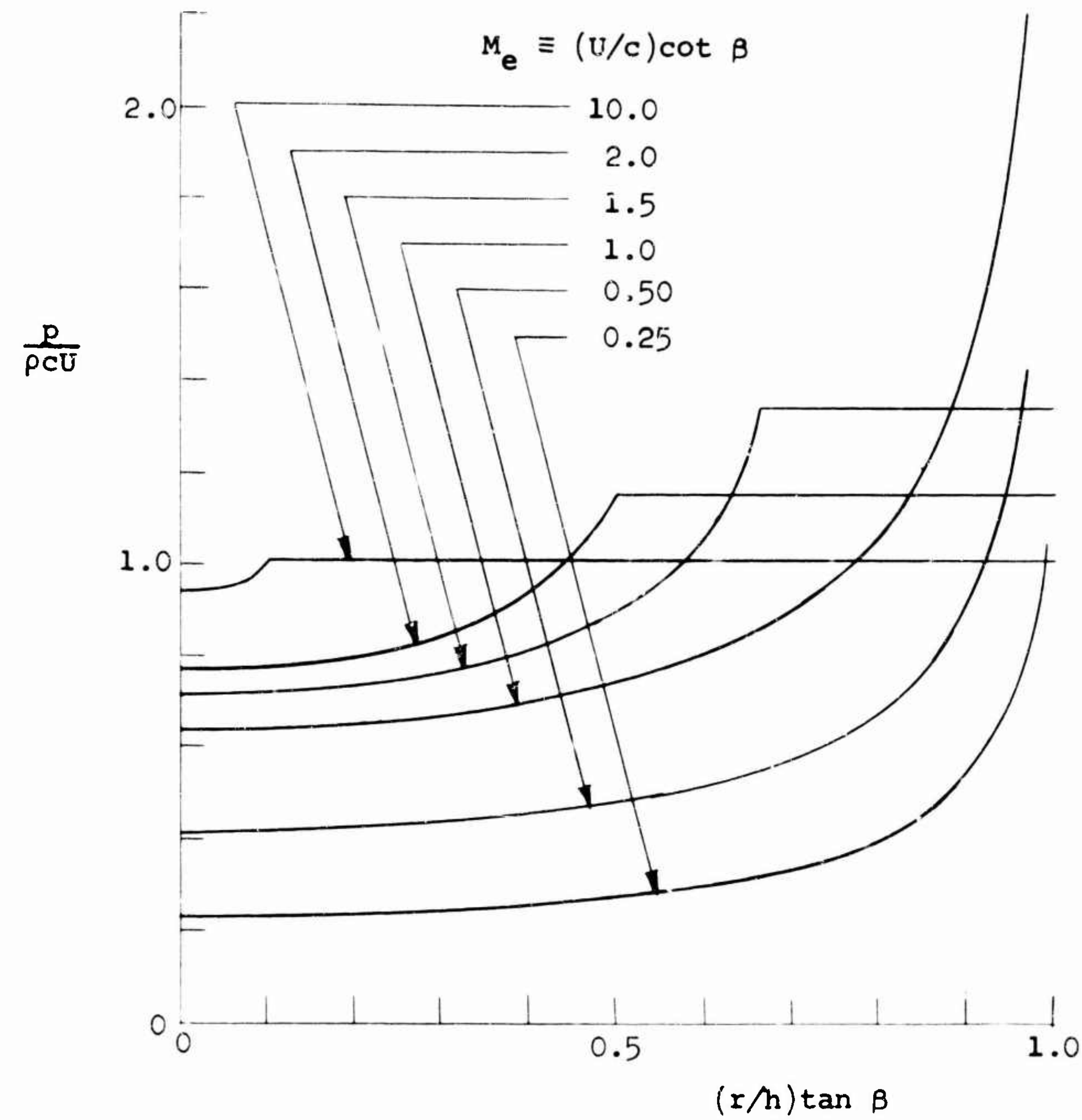


FIGURE 23

PRESSURE DISTRIBUTION ON WEDGE ENTERING COMPRESSIBLE WATER (Skalak & Feit 1963)

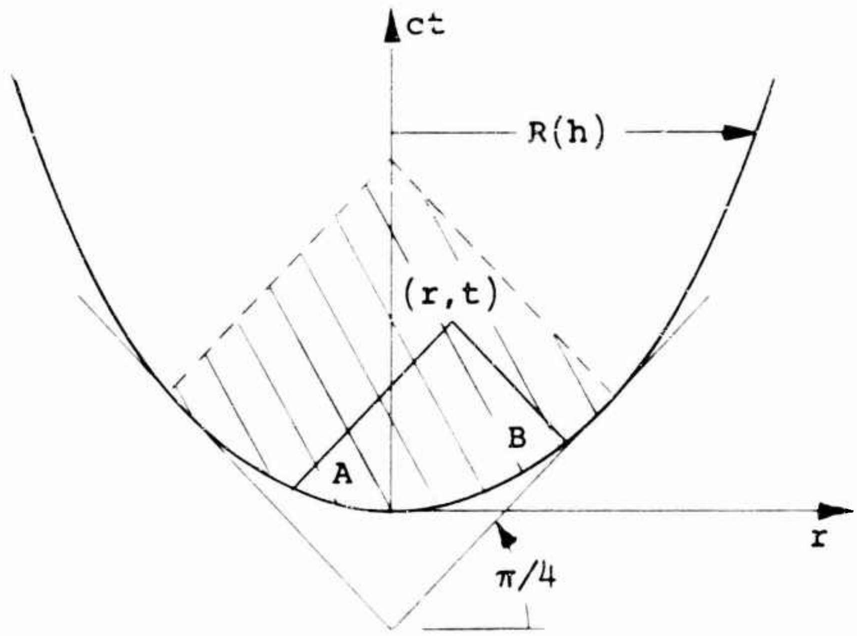


FIGURE 24

REGION WHERE PRESSURE ON ARBITRARILY SHAPED BODY IN COMPRESSIBLE WATER ENTRY IS RELATIVELY EASY TO CALCULATE

$$k = \left[1 - \frac{U^2 \mathrm{ctn}^2 \beta}{c^2}\right]^{\frac{1}{2}} \tag{95}$$

The treatment of non-polygonal shapes with this formulation is more difficult, and no results have been published. Trilling (1950a) suggests approximating the profile by a polygonal one, but this seems rather drastic. The corresponding problem in wing theory is reduced by the Evvard-Krasilschikova theory (see Jones & Cohen 1960) to the evaluation of a few definite integrals which are, however, not readily expressible in terms of tabulated functions. The simplest case is where the Mach cone facing upstream from the point under study intersects the body profile upstream of those points where the edge Mach number becomes sonic, as shown in Fig. 24. For such points, the pressure can be expressed as (cf. equation (13.16) in Jones & Cohen 1960)

$$\frac{p}{\rho c U} = \frac{1}{\pi} \int_{r_B(r,t)}^{r_A(r,t)} \left[\{ct - \frac{c}{U} X(\eta)\}^2 - (r-\eta)^2 \right]^{-\frac{1}{2}} d\eta \tag{96}$$

where, as shown in Fig. 24, A and B are the points at which the Mach forecone from (r,t) intersects the body contour in the r-ct plane.

For example, consider the constant-speed entry of a parabolic cylinder of nose radius ℓ , for which

$$X(r) = \frac{1}{2} r^2/\ell \tag{97}$$

Substituting (97) into (96), we find that the pressure can be expressed in terms of an elliptic integral of the first kind, since the integrand is simply the inverse square root of a quartic polynomial (Byrd & Friedman 1954). However, to actually write out the result, we would have to solve the quartic, which turns out to be quite messy. The task simplifies considerably when $r = 0$, and so we obtain for the stagnation-point pressure the formula

$$\frac{1}{\rho c U} p(-h,0,t) = \frac{4}{\pi} \left[1 + \sqrt{1 + \frac{2ct}{\ell}} \right]^{-1} K(k) \tag{98}$$

where the modulus of the complete elliptic integral is

$$k = \frac{\sqrt{1+2ct/\ell} - 1}{\sqrt{1+2ct/\ell} + 1} \tag{99}$$

These results are good for $t \leq \frac{3}{2} \ell U/c^2$, at which time the forecone from the stagnation point intersects the sonic points on the profile.

The entry of three-dimensional shapes into compressible water offers difficulties of the same order of magnitude as those encountered in treating curvilinear two-dimensional profiles. Trilling (1950a) treated slender cones and flat-faced circular cylinders (entering end on), but his analysis has been criticized by Korkegi (1952), who used retarded-potential concepts to study the entry of a paraboloid of revolution for times in which the only edges involved are supersonic (cf. our treatment above of the corresponding two-dimensional case). Pressure distributions on the paraboloid were calculated, but were not integrated to give the history of the force, since, like the two-dimensional results, they involve an elliptic integral.

It is of interest to note, as was pointed out by Korkegi, that when the edge Mach number is supersonic, the free surface is truly undisturbed. Thus, in this case, the linearized solution exactly satisfies the free-surface boundary conditions. Of course, the theory is still only approximate, since the continuity equation has been linearized in deriving the wave equation (88), and the body boundary condition has also been approximated. Nevertheless, it is of interest to note that the wetting and free-surface corrections discussed in § 2.3 are identically zero when the lateral expansion rate of the points at which the body intercepts the undisturbed free surface is supersonic. When the edge Mach number is subsonic, however, the above method of estimating compressibility effects is no further advanced than von Kármán's (1929) approximation to the incompressible flow solution.

Borg (1959,1960) has considered wedge and cone impact with a somewhat different formulation. In essence, the entering body is fitted with a shape of semicircular cross-section, which suggests a restriction on the deadrise angle to 45°. A particularly simple assumption regarding the (purely radial) velocity field is found to be satisfied when the entry speed is $c/2$ in the wedge case and $2c/3$ in the cone case. Further approximations are made in an attempt to generalize the analysis to other body shapes and to account for the nonlinearities in the free-surface boundary conditions, but no very convincing results are obtained.

Another, more plausible, formulation has been given recently by Rhyming (1963), who develops a slender-body theory for the entry of bodies of revolution. Thus, supersonic sources are distributed on the axis of motion, with their strength being antisymmetric about the undisturbed free surface. The retarded-potential concept is used to write the solution as

$$\phi(x,r,t) = -\frac{1}{4\pi}\int_{-h}^{h}\frac{\sigma(\xi,t-[(x-\xi)^2+r^2]^{\frac{1}{2}}/c)}{[(x-\xi)^2+r^2]^{\frac{1}{2}}}\,d\xi \tag{100}$$

where

$$\begin{aligned}\sigma(x,t) &= \frac{\partial}{\partial t}R^2(x+h) \qquad \text{for} \quad x < 0\\ &= -\sigma(-x,t) \qquad \text{for} \quad x > 0\end{aligned} \tag{101}$$

Curiously, no Mach number effects on the pressure distribution on a cone are found.

References to recent Russian work on compressible entry are given by Wehausen (1963).

3.4 Effects of Air Density

Inclusion of compressibility effects is seen to remove some of the more glaring defects of the formulation of § 1.1. The impact pressure is rendered finite, and, at least when the edge Mach number is supersonic, even the linearized solution is free of spurious singularities, and so seems a reliable approximation.

Nevertheless, one should not regard water compressibility as a panacea for the difficulties of the conventional solution. To do so would be to claim that compressibility is important in the entry of any blunt- or round-nosed body, whatever the entry speed. This does not seem physically plausible for very low entry speeds.

It is, therefore, of interest to note that the impact pressure predicted by incompressible-flow theory is finite if density of the air is taken into account (Moran & Kerney 1964). To see this, let us consider the changes required in the conventional formulation when the air density is finite, but the water is still regarded as incompressible.

As noted in § 1.2, we must anticipate a discontinuity in the potential and its derivatives across the air-water interface. Thus, let ϕ^+ and ϕ^- be the potentials above and below the free surface, respectively, so that

$$\phi = \phi^+ \qquad \text{for} \quad x > \Delta(r,t)$$

$$\phi = \phi^- \qquad \text{for} \quad x < \Delta(r,t) \tag{102}$$

Similarly, let ρ^+ and ρ^- be the densities of the air and of the water, respectively, and let

$$\delta \equiv \rho^+/\rho^- \tag{103}$$

Note that $\delta \approx 1/800$.

Laplace's equation (4) still applies throughout the flow field, as does Bernoulli's equation (5). Also, the body boundary condition (8) is unchanged. However, since the pressure above the free surface is no longer necessarily constant, the dynamical free-surface boundary condition becomes, in the absence of gravity,

$$\rho^{+}\left[\phi_t^{+} + \frac{1}{2}(\nabla\phi^{+})^2\right] = \rho^{-}\left[\phi_t^{-} + \frac{1}{2}(\nabla\phi^{-})^2\right] \qquad \text{on} \quad x = \Delta(r,t) \tag{104}$$

That is, the pressure is required to be continuous across the free surface. The kinematical free-surface boundary condition (6) still holds, but we also require continuity of the velocity component normal to the free surface across that interface, or

$$\phi_x^{+} - \phi_r^{+}\Delta_r = \phi_x^{-} - \phi_r^{-}\Delta_r \qquad \text{on} \quad x = \Delta(r,t) \tag{105}$$

It is this last requirement which is responsible for the removal of the singularity in the impact pressure found in the zero-air-density approximation. Consider, for simplicity, the pre-impact phase of the vertical symmetric entry of a blunt-nosed body of revolution. From the kinematic body and free-surface boundary conditions, we expect that the magnitude of the fluid velocity along the x-axis decreases monotonically and continuously from the entry speed U immediately below the body nose to zero infinitely far below the free surface. Conversely, at a fixed distance along the x-axis from the air-water interface, $|u(x,0,t)|$ may be expected to increase continuously from 0 to U as that point is approached by the body's stagnation point. Then, at the moment of impact, the free surface at the impact point has already been accelerated to the body speed. The inclusion of air-density effects, therefore, eliminates the abrupt velocity change at impact which is clearly responsible for the infinite impact pressure found in the conventional formulation.

This is not to say that compressibility effects are always negligible. Certainly, if the body speed is high enough, such effects will be important. Indeed, shock waves are observed when the entry speed is high enough (McMillen, et al, 1950). However, it is certain that the neglect of air density exaggerates the importance of compressibility, and that the assumption of incompressibility is tenable when the entry speed is low enough. Since, according to an incompressible-flow analysis, the fluid speeds are proportional to the entry speed, the results of such an analysis could be used to determine an upper limit on the entry speed below which the phenomena could be treated as incompressible. Consequently, in cases where the infinite impact pressure predicted by standard theory is unacceptable, but where the entry speed is quite small compared to the sound speed, it would seem advisable to relax the zero-air-density assumption before worrying about the compressibility of water.

Unfortunately, the air-density effects described above are important only during that phase of the motion in which non-linearities in the free-surface boundary conditions are least manageable. Thus far, it has been found necessary to make small-disturbance assumptions in order to obtain analytical results. Moran (1964a) has treated the linearized problem of the vertical constant-speed approach to the surface of a point source of constant strength at arbitrary Froude number, while Moran & Kerney (1964) have solved the infinite-Froude-number vertical-exit and -entry problems for slender bodies of revolution to second order in the body thickness ratio. While these analyses are thus not directly applicable to the impact-pressure problem which prompted us to look into air-density effects, they are pertinent to a numerical solution which could solve this problem, as will be discussed shortly. Thus, for purposes of illustration, the linearized infinite-Froude-number solution corresponding to the motion of a point source near the surface will be reproduced here.

The linearization of the free-surface boundary conditions (104) and (105) proceeds exactly as described in § 2.1 for the zero-air-density case, and leads to the requirements

$$\phi^- = \delta\, \phi^+ \qquad \text{on} \quad x = 0 \tag{106}$$

$$\phi_x^- = \phi_x^+ \qquad \text{on} \quad x = 0 \tag{107}$$

We let

$$\phi = \phi_S + \phi_V \tag{108}$$

where ϕ_S is the potential of the given source

$$\phi_S = -\frac{Q}{4\pi}\left[(x-x_o)^2 + r^2\right]^{-\frac{1}{2}} \tag{109}$$

and ϕ_V is to be determined so that ϕ satisfies the free-surface boundary conditions and the boundary condition of no disturbance at infinity. Since ϕ_S satisfies this last requirement by itself, we have

$$\phi_V \to 0 \qquad \text{as} \quad x^2+r^2 \to \infty \tag{110}$$

Also, ϕ_V must be regular throughout the fluid; it can be singular only on the free surface.

Equation (107) is automatically satisfied if we let ϕ_V be the potential of a vortex sheet on $x = 0$, and so implies that

$$\phi_V^+ = -\phi_V^- \qquad \text{on} \quad x = 0 \tag{111}$$

From (106), (108), and (111), we obtain

$$\phi_V^+ = \frac{1-\delta}{1+\delta}\,\phi_S \qquad \text{on} \quad x = 0 \tag{112}$$

$$\phi_V^- = -\frac{1-\delta}{1+\delta}\,\phi_S \qquad \text{on} \quad x = 0 \tag{113}$$

These relations determine the strength of the vortex sheet on $x = 0$ directly. However, we can also use them to express the solution more simply in terms of point singularities.

Suppose, for example, that the given source is located above the surface; i.e., $x_o > 0$. Then both ϕ_S and ϕ_V^- are free from singularities in $x < 0$, and the function

$$\varphi \equiv \phi_V^- + \frac{1-\delta}{1+\delta}\,\phi_S \tag{114}$$

satisfies Laplace's equation everywhere in the region bounded by the plane surface $\Sigma_P = \{x=0,\ r<\mathcal{R}\}$ and Σ_H , the lower half of a sphere of radius $\mathcal{R}$ and centered at $(0,0)$. Then, from Green's theorem,

$$\int_W (\nabla\varphi)^2\, d\tau = \int_{\Sigma_P+\Sigma_H} \varphi\,\frac{\partial\varphi}{\partial n}\, d\sigma \tag{115}$$

where W is the volume bounded by Σ_P and Σ_H , and n is the normal directed outward from the bounding surfaces. Assuming that ϕ_V^- is source-like far from the origin, as can be verified <u>a posteriori</u>, we see that the integral in (115) vanishes as $\mathcal{R} \to \infty$. Since, from (113) and (114), $\varphi = 0$ on Σ_P , equation (115) shows that φ is constant in W , viz., again from (113), zero.

In other words, (113) can be continued off the plane $x = 0$ into the region below the surface, and we get

$$\phi_V^- = -\frac{1}{4\pi}\,\frac{1-\delta}{1+\delta}\,Q\,\{(x-x_o)^2 + r^2\}^{-\frac{1}{2}} \tag{116}$$

Though this is singular at $(x_o,0)$, from the definition (102) $\phi = \phi^-$ only in the region below the surface, and so the requirement that ϕ_V be regular off the plane $x = 0$ is not violated by (106).

To complete the solution, we need ϕ_V^+ . From the symmetry of (111), and the requirement that ϕ_V^+ be regular in $x > 0$, we see that

$$\phi_V^+ = \frac{1}{4\pi} \frac{1-\delta}{1+\delta} Q \, \{(x+x_o)^2 + r^2\}^{-\frac{1}{2}} \tag{117}$$

Thus, given a source of strength Q in the air, the flow above the free surface is due to the original source plus a submerged sink of strength $-\left[(1-\delta)/(1+\delta)\right]Q$ at its image in the undisturbed surface, while the flow below is due to the original source plus one of strength $\left[(1-\delta)/(1+\delta)\right]Q$ which is coincident with the original source.*

Now letting $x_o = -h$, we plug (108), (109), and (116) into the linearized kinematic boundary condition (26) and integrate to obtain the free-surface shape;

$$\Delta(r,t) = -\frac{1}{2\pi U} \frac{Q\delta}{1+\delta} \{x_o^2 + r^2\}^{-\frac{1}{2}} \tag{118}$$

This result and those which preceded it are not valid when the source gets too close to the surface; certainly, they must

*This solution is highly reminiscent of that for the electrostatic problem in which a point charge is positioned near a plane discontinuity in the dielectric constant (Mason & Weaver).

break down when the image singularity hits the depressed free surface. From (118), the depression is proportional to δ , and so may be estimated in the entry of a round-nosed body as being of the order of $\delta \ell$, where ℓ is the nose radius of curvature. Thus we estimate the duration of significant air-density effects as

$$\frac{U\tau_A}{\ell} = O(\delta) \tag{119}$$

For comparison, we estimate the duration of compressibility effects as the time required for the edge Mach number to become sonic, and so obtain

$$\frac{U\tau_C}{\ell} = O(U^2/c^2) \tag{120}$$

These relations bear out our argument that air density effects are more significant than compressibility effects at low entry speeds.*

Since, as noted above, the effects of air density are important only when they are least amenable to analytic attack, it is of interest to consider the formulation of a numerical solution of the surface-crossing problem in which the air density

---

*An estimate similar to (120) was made by Chu (1960), who also concluded that the duration of significant compressibility effects is negligible on the time scale of interest in ship slamming, for example.

is regarded as finite. We recommend that such a solution be carried out in terms of singularity distributions, in which both the body surface and the free water surface are covered with vortex sheets. This would automatically satisfy the Laplace equation (and so would avoid the large consumption of time and machine storage space which are characteristic of relaxation methods), the boundary condition at infinity, and the kinematic boundary condition of continuous normal velocity across the free surface. Also, use of vortices rather than sources or doublets on the body surface simplifies the body boundary condition; compare Smith & Pierce's (1958) use of source distributions with Landweber's (1951, 1959) vortex-distribution solution.

For numerical purposes, the free-surface and the body-surface profiles may be approximated by a contour composed of a specified number of connected straight-line segments, on each of which the vortex strength is taken to be constant. Such an approximation was found by Smith & Pierce to be more than adequate in unbounded flow provided, of course, that a sufficient number of approximating elements are used. The junctures of the segments representing the free surface ought to represent definite fluid particles, whose trajectories would be calculated from a finite-difference version of the kinematic free-surface boundary condition. The strengths of the vortices on the elements should then be calculated from a finite-difference version of the dynamical free-surface boundary condition. Once these are known, the integral equation corresponding to the body boundary condition can be approximated by a set of algebraic equations for the vortex strengths on each of the elements representing the body and solved. To improve the accuracy of the finite-difference approximation to the time derivatives in the free-surface boundary conditions, an iteration procedure based on forward and backward differencing could be set up.

One of the more difficult problems in the numerical solution would be the description, during the broach phase of the motion, of the absorption of free-surface particles by the body surface. Perhaps this can be done by concentrating the juncture points describing the surface shape near the axis of motion, and taking the time interval so that, at the end of the interval, the particle which had been adjacent to the particle at the contact point at the beginning of the interval moves to the juncture of the body and free surfaces (as suggested by Chu 1960). In this process a certain amount of freedom as to the disposition of the juncture points on the body surface is available — i.e., the contact point can be forced to be a point at which two of the line segments representing the body profile meet, as is convenient — since the body boundary condition does not involve time explicitly.

Similar problems occur, for example, towards the end of the broach phase of the entry problem, in which fluid particles on the body surface transfer to the free surface. If the resolution of the "reemission" problem turns out to be more difficult than that of the "absorption" problem, advantage can be taken of the reversibility of the flow (Moran & Kerney 1964) to work the problem partly as a water-exit problem and partly as one of entry.

Although errors in the spray region are probably unavoidable, it may be noted that the exact boundary conditions require that the free-surface vortex strength vanish at the contact point, since the fluid velocity must be parallel to the body surface on both sides of the contact point and since the component of fluid velocity normal to the free surface is continuous through the contact point. Thus numerical errors in locating the contact point can be minimized if the vortex strength is specified as zero on the element of the free surface adjacent to the body surface. In any case, such errors are probably not too serious in so far as computing the body-surface pressure distribution is concerned, since the thinness of the spray demands that the pressure be nearly constant thru the spray, and hence the pressure on the portion of the body wetted by the spray is nearly zero.

To start the solution at a time when the body is a finite distance from the surface, the linearized solution described above can be used. Since a second-order solution is available (Moran & Kerney 1964), it is possible to check how close the body can approach the surface before the linearization of the free-surface boundary conditions loses validity (cf. § 2.6). During the complete-submergence phase, this critical depth can be calculated without taking air density into account without significant error (but with considerable reduction in computation). As for the airborne phase, the distance between the body and the surface, above which the linearized results can be used, can be set equal to the aforementioned critical depth. This would seem to be on the conservative side, and again would minimize the computational effort required.

Finally, we note that the free surface is only slightly disturbed far from the body even during broach. Thus the free-surface boundary conditions for r sufficiently large ought to be satisfied by the linearized results even in a numerical solution so as to reduce the computation time and storage requirements.

SUMMARY AND CONCLUSIONS

It is seen from the preceding survey that, despite a substantial expenditure of effort, there exists no reliable means for predicting the loads felt during the surface crossing. This failure reflects the mathematical nature of the problem. Under standard assumptions, one must satisfy nonlinear boundary conditions on a surface whose shape is unknown _a priori_. Such difficulties can be avoided by linearization, which, however, leads to an unacceptably fictitious singlarity at the intersection of the body and the undisturbed position of the free surface. Other approximations which have led to analytic solutions are also non-uniformly valid.

Attempts to treat the problem numerically have met with little more success. The only solutions available refer to the conical-flow problems of wedge and cone entry, and of these all but Borg's (1957) study of a single case contain serious errors. Treatment of arbitrarily shaped bodies offers even greater difficulties, the more so because the pressure felt at impact by a round-nosed body is theoretically infinite under the "conventional" set of assumptions.

We nevertheless recommend that further effort be devoted to the development of numerical methods for solving water-exit and -entry problems. Specifically, for conical-flow problems, Hillman's (1946) approach ought to be corrected as explained in § 2.5.1 and programmed for a computer, so that a set of reliable solutions for wedge and cone entry could be obtained. Such solutions would be of interest in themselves as well as serving as standards of comparison for more approximate analyses.

The development of a computer code capable of handling non-conical problems offers greater difficulties, partly because the time dependence is more easily handled in conical situations, but mostly because of the impact singularity found in the surface crossing of blunt bodies under the conventional formulation. To eliminate this singularity, it is suggested in § 3.4 that, except in high-speed entry, the usual assumption of incompressible flow may be retained, but the finite density of the air should be ta into account. Specific recommendations as to how this can be done are also detailed in § 3.4.

To these recommendations on numerical solutions of surface-crossing problems we should add a plea that cavitation be taken into account. Unfortunately, it is difficult to see at present how this might be done satisfactorily. On the other hand, since the problem in the absence of cavitation seems to require a numerical solution, there is no reason to believe that cavitating surface-crossing can be adequately handled analytically. Thus the carrying out of the present recommendations may well be a necessary first step towards the development of a theory of water-exit and -entry in which cavitation is accounted for numerically.

REFERENCES

ARMSTRONG, A. H., & DODD, K. N. 1957 The virtual mass of a double cone using the spindle approximation. Armament Research & Development Establishment, Memorandum (B) 6/57.

BEHRBOHM, H. 1952 The lifting trapezoidal wing with small aspect ratio at supersonic speed. SAAB Aircraft Co., Tech. Note 10 (see also Jones & Cohen 1960).

BISPLINGHOFF, R. L., & DOHERTY, C. S. 1952 Some studies of the impact of vee wedges on a water surface. J. Franklin Institute, 253, 547-561.

BORG, S. F. 1957 Some contributions to the wedge-water entry problem. J. Eng. Mech. Div., Proc. Amer. Soc. Civil Eng., 83 (EM2), paper 1214.

BORG, S. F. 1959 Initial wedge impact on a compressible fluid. J. Appl. Phys., 30, 1432-1436.

BORG, S. F. 1960 Three-dimensional wedge impact on a compressible fluid. J. Appl. Phys., 31, 438-439.

BRARD, R. 1948 Introduction à l'étude théorique du tangage en marche. Bull. Assoc. Tech. Mar. Aéro. 47, 455-471.

BRESLIN, J. P. 1958 (Confidential title). Westinghouse Corp., Pittsburgh, Rept. ASE #481.

BRESLIN, J. P., & KAPLAN, P. 1957 Theoretical analysis of hydrodynamic effects on missiles approaching the free surface, including the influence of waves. Davidson Lab., Stevens Inst. Tech., Note 444.

BYRD, P. F., & FRIEDMAN, M. D. 1954 Handbook of Elliptic Integrals for Engineers and Physicists. Berlin: Springer-Verlag.

CHOU, P. Y. 1946 On impact of spheres upon water. Water Entry and Underwater Ballistics of Projectiles, Chap. 8. Cal. Inst. Tech., OSRD Rept. 2551.

CHU, W.-H. 1960 On the development of a more accurate method for calculating body-water impact pressures. Southwest Res. Inst., Tech. Rept. No. 2.

CHU, W.-H., & ABRAMSON, H. N. 1959 Hydrodynamic theories of ship slamming - review and extension. Southwest Res. Inst. Tech. Rept. No. 1. Also, J. Ship Res. 4, No. 4, 9-21 (1961).

CHU, W.-H., & FALCONER, D. R. 1963 Further development of a more accurate method for calculating body-water impact pressures. Southwest Res. Inst. Tech. Rept. No. 5.

CLARK, M. E., & ROBERTSON, J. M. 1960 A study of the kinematics of buoyant-body water exit, part II. Vertical exit results. U. Illinois Dept. Theo. Appl. Mech., Rept. No. 181.

COOMBS, A. 1956 The oblique water entry of a cone. Armament Research and Development Estab., (see Cox & Maccoll 1957).

COOPER, E. P. 1949 Theory of water entry of missiles with flat noses. Naval Ordnance Test Station, NAVORD Rept. 1154 (NOTS 208).

COX, R. N., & MACCOLL, J. W. 1957 Recent contributions to basic hydroballistics. (First) Symposium on Naval Hydrodynamics, Nat. Acad. Sci.-Nat. Res. Com. Publication 515, 215-240.

CUMBERBATCH, E. 1958 Two-dimensional planing at high Froude number. J. Fluid Mech. 4, 466-478.

CUTHBERT, J. W., & KERR, K. P. 1962 The effect of waves on the ideal fluid forces on a slender axisymmetric body having three degrees of freedom. Lockheed Missiles & Space Co., T.M. 81-73-15.

DERGARABEDIAN, P. 1955 Compressibility effects during water entry. Naval Ordnance Test Station, NAVORD Rept. 3523 (NOTS 1159).

EGOROV, I. T. 1956 Impact on a compressible fluid. Prik. Mate. Mekh. 20, 67-72 (translated by M. L. Mahler in NACA TM 1413, 1958).

FABULA, A. G. 1955 Some theoretical calculations on the impact of circular cylinders on water. Naval Ordnance Test Station Tech. Prog. Rept. 111 (NOTS 929).

FABULA, A. G. 1957 Ellipse-fitting approximation of two-dimensional normal symmetric impact of rigid bodies on water. Proc. Fifth Midwestern Conf. on Fluid Mech., 299-315. Ann Arbor: U. Michigan Press.

FABULA, A. G., & RUGGLES, I. D. 1955 Vertical broadside water impact of circular cylinder: growing circular-arc approximation. Naval Ordnance Test Station, NAVORD Rept. 4947 (NOTS 1268).

FRAENKEL, L. E. 1958a Incompressible flow past quasi-cylindrical bodies and some associated problems. Q. J. Mech. Appl. Math. **11**, 212-222.

FRAENKEL, L. E. 1958b On the vertical water entry of a slender cone. Armament Research and Development Estab. Memo. (B) 22/58.

GARABEDIAN, P. R. 1953 Oblique water entry of a wedge. Comm. Pure Appl. Math. **6**, 157-165.

GOODMAN, T. R. 1960 Lateral hydrodynamic forces on an airborne missile launched underwater. ARS J. **30**, 1095-1096.

GREEN, A. E. 1935 The gliding of a plate on a stream of finite depth. Proc. Camb. Phil. Soc. **31**, 589-603.

GREEN, A. E. 1936a The gliding of a plate on a stream of finite depth. Part II. Proc. Camb. Phil. Soc. **32**, 67-85, 183.

GREEN, A. E. 1936b Note on the gliding of a plate on the surface of a stream. Proc. Camb. Phil. Soc. **32**, 248-252.

GUNN, J. C. 1947 Linearized supersonic airfoil theory. Parts I and II Phil. Trans. Roy. Soc. London, **240A**, 327-373.

HASKIND, M. D. 1946 The oscillations of a ship in still water. Izv. Akad. Nauk SSSR Otd. Tekhn. Nauk **1946**, 23. Translated in Soc. Nav. Arch. Mar. Eng. Tech. & Res. Bull. 1-12 (1953).

HILLMAN, A. 1946 Vertical impact on a water surface by a 60° cone. Math. Tables Project, New York U. (see Shiffman & Spencer 1951).

JONES, R. T., & COHEN, D. 1960 High Speed Wing Theory. Princeton U. Press.

KARZAS, W. J. 1952 The vertical water entry of a wedge. Naval Ordnance Test Station, Tech. Memo. No. 576.

KORKEGI, R. H. 1952 Pressure distribution on a sphere due to vertical impact on a plane water surface. Naval Ordnance Test Station, Tech. Memo No. 656 (see also Dergarabedian 1955).

LAGERSTROM, P. A., & GRAHAM, M. E. 1947 Low aspect ratio rectangular wings in supersonic flow. Douglas Aircraft Co. Rept. SM-13110 (see also Lagerstrom & Graham 1951).

LAGERSTROM, P. A., & GRAHAM, M. E. 1951 Remarks on low-aspect-ratio configurations in supersonic flow. J. Aero. Sci. **18**, 91-96, 100.

LAMB, H. 1913 On some cases of wave-motion on deep water. Ann. de Mat. **21**, 237-250. (see also Lamb 1945, § 247).

LAMB, H. 1945 Hydrodynamics, 6th ed. New York: Dover Pub.

LANDWEBER, L. 1951 The axially symmetric potential flow about elongated bodies of revolution. D. W. Taylor Model Basin Rept. 761.

LANDWEBER, L. 1959 Potential flow about bodies of revolution and symmetric two-dimensional forms. Iowa Inst. Hyd. Res. Contract Nonr-1611(01).

LANDWEBER, L., & YIH, C.-S. 1956 Forces, moments, and added masses for Rankine bodies. J. Fluid Mech. **1**, 319-336.

LEWIS, F. M. 1929 The inertia of the water surrounding a vibrating ship. Soc. Nav. Arch. Mar. Eng. Trans. **37**, 1-20.

MACKIE, A. G. 1962 A linearized theory of the water entry problem. Quart. J. Mech. Appl. Math. **15**, 137-151.

MACKIE, A. G. 1963 Initial value problems in water wave theory. Australian Math. Soc. J. **3**, 340-350.

MARTIN, M. 1960 Preliminary study of free surface effects on the water to air exit of a slender body. Stevens Inst. Tech. Davidson Lab. Tech. Memo. No. 125.

MASON, M., & WEAVER, W. (no date) The Electromagnetic Field. New York: Dover Pub.

MCMILLEN, J. H., KRAMER, R. L., & ALLMAND, D. E. 1950 Shadowgraphs of spherical missiles entering water at supersonic speeds. J. Appl. Phys. 21, 1341-1342.

OGILVIE, T. F. 1962 Ship slamming and supersonic airfoil flow. Office of Naval Res. Tech Rept. ONRL-76-62.

PIERSON, J. D. 1948 On the pressure distribution for a wedge penetrating a fluid surface. Stevens Inst. Tech. Exp. Towing Tank Rept. 336.

PIERSON, J. D. 1950 The penetration of a fluid surface by a wedge. Stevens Inst. Tech. Exp. Towing Tank Rept. 381 (see also Pierson 1951).

PIERSON, J. D. 1951 On the virtual mass of water associated with an immersing wedge. J. Aero. Sci. **18**, 430-431.

PLESSET, M. S. 1942 An investigation of the impact force on torpedoes entering water. Douglas Aircraft Co. Rept. No. SM 3937.

RYHMING, I. L. 1963 Compressibility effects of slender bodies entering vertically into water. AIAA J. **1**, 702-703.

RICH, E. J., & KARP, S.N. 1960 The virtual mass of a finite conical shell. New York U. Inst. of Math. Sci. Rept. AFCRC-TN-60-393.

SAKAI, T., HUSIMI, Y., & HATOYAMA, M. 1933 On the resistance experienced by a body moving in an incompressible perfect fluid towards its free surface. Proc. Physico-Math. Soc. Japan **15** (3rd series), 4-11.

SCHNITZER, E., & HATHAWAY, M.E. 1953 Estimation of hydrodynamic impact loads and pressure distributions on bodies approximating elliptical cylinders with special references to water landings of helicopters. Nat. Adv. Comm. Aero. Tech. Note 2889.

SHIFFMAN, M. & SPENCER, D. C. 1945a The force of impact on a sphere striking a water surface. Approximation by the flow about a lens. Appl. Math. Panel Rept. 42.1R, Appl. Math. Group-New York U. No. 105.

SHIFFMAN, M. & SPENCER, D. C. 1945b The force of impact on a sphere striking a water surface. Second approximation. Appl. Math. Panel Rept. 42.2R, Appl. Math. Group-New York U. No. 133

SHIFFMAN, M., & SPENCER, D. C. 1947 The flow of an ideal incompressible fluid about a lens. Quart. Appl. Math. **5**, 270-288.

MILNE-THOMSON, L. M. 1960 Theoretical Hydrodynamics, 4th ed. New York: Macmillan.

MONAGHAN, R. J. 1949 A theoretical examination of the effect of deadrise on wetted area and associated mass in seaplane-water impacts. Roy. Aero. Estab. Tech. Note Aero. 1989. Also, Aero. Res.Coun. Rept. & Memo. No. 2681 (1952).

MORAN, J. P. 1961 The vertical water-exit and -entry of slender symmetric bodies. J. Aerospace Sci. 28, 803-812.

MORAN, J. P. 1962a Line source distributions in axisymmetric incompressible flow. II. Solution of the indirect problem. Therm Adv. Res. Rept. TAR-TR 621.

MORAN, J. P. 1962b Erratum: An equivalence principle for water-exit and -entry problems. J. Aerospace Sci. 29, 1007-1008.

MORAN, J. P. 1963 Line source distributions and slender body theory. J. Fluid Mech. 17, 285-304.

MORAN, J. P. 1964a Image solution for vertical motion of a point source towards a free surface. J. Fluid Mech. 18, 315-320.

MORAN, J. P. 1964b Addendum: The vertical water-exit and -entry of slender symmetric bodies. AIAA J. 2, 1480-1482.

MORAN, J. P., & KERNEY, K. P. 1964 On the small-perturbation theory of water-exit and -entry. Developments in Mechanics (S. Ostrach, ed.), 476-504. Oxford: Pergamon Press.

NELSON, D. M. 1961 Forces on a slender body of revolution in arbitrary planar motion in a train of gravity waves. Naval Ordnance Test Station, NAVWEPS Rept. 7767 (NOTS TP 2746).

NISEWANGER, C. R. 1961 Experimental determination of pressure distribution on a sphere during water entry. Naval Ordnance Test Station, NAVWEPS Rept. 7808 (NOTS TP 2806).

OCHI, K. M., & BLEDSOE, M. D. 1962 Hydrodynamic impact with application to ship slamming. Fourth Symp. Naval Hydro., 867-912. Office of Naval Res., ACR-73.

SHIFFMAN, M., & SPENCER, D. C. 1951 The force of impact on a cone striking a water surface (vertical entry). Comm. Pure & Appl. Math. 4, 379-417.

SKALAK, R., & FEIT, D. 1963 Impact on the surface of a compressible fluid. Columbia U. Dept. of Civil Eng. & Eng. Mech. Tech. Rept. No. 31.

SMITH, A.M.O., & PIERCE, J. 1958 Exact solution of the Neumann problem. Calculation of non-circulatory plane and axially symmetric flows about or within arbitrary boundaries. Proc. Third U. S. Nat. Cong. of Appl. Mech., 807-815. New York: Amer. Soc. Mech. Eng.

STEWARTSON, K. 1950 Supersonic flow over an inclined wing of zero aspect ratio. Proc. Camb. Phil. Soc. 46, 307-315.

SZEBEHELY, V. G. 1959 Hydrodynamic impact. Appl Mech. Rev. 12, 297-300.

TAYLOR, J. L. 1930 Some hydrodynamical inertia coefficients. Phil. Mag. 9 (series 7), 161-183.

TRILLING, L. 1950a The compression shock due to the impact of a body on a plane water surface. Naval Ordnance Test Station Tech. Memo. 808-39.

TRILLING, L. 1950b The impact of a body on a water surface at an arbitrary angle. J. Appl. Phys. 21, 161-170.

VON KÁRMÁN, TH. 1929 The impact of seaplane floats during landing. Nat. Adv. Comm. Aero. Tech. Note 321.

WAGNER, H. 1931 Landing of seaplanes. Zeitschrift für Flugtechnik und Motorluftschiffahrt 22, 1-8. Translated by J. Vanier as NACA TM 622 (1931).

WAGNER, H. 1932 Über Stoss- und Gleitvorgänge an der Oberflächen von Flüssigkeiten. Zeitschrift für angewandten Mathematik and Mechanik 12, 193-215. (see also Wagner 1931 and Wagner 1950).

WAGNER, H. 1950 Impact forces at water entry. Proc. Eighth Underwater Ballistics Conf., 3-16. Iowa Inst. Hyd. Res.

WARD, G. N. 1955 Linearized Theory of Steady High Speed Flow, Cambridge U. Press.

WEHAUSEN, J. V. 1963 Recent developments in free-surface flows. U. Calif. Inst. of Eng. Res. Rept. NA-63-5.

WEHAUSEN, J. V., & LAITONE, E. V. 1960 Surface waves. Handbuch der Physik (S. Flugge, ed.), **9**, 446-778. Berlin: Springer-Verlag.

WOO, T. C. 1959 The motion of a submerged body moving vertically towards a free surface. Brown U. Div. of Appl. Math.

YU, Y.-T. 1945 Virtual masses of rectangular plates and parallelipipeds in water. J. Appl. Phys. **16**, 724-730.